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ANALYTICAL STUDY OF THE TEVATRON NONLINEAR DYNAMICS

F. Willeke

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Abstract

The dependence of the TEVATRON dynamic aperture on the systematic high field multipole errors of the bending magnets is studied using a distortion function technique including second order effects in perturbation theory. The results are in good agreement with tracking studies. It can be concluded that the dynamic aperture of the TEVATRON is given by the break off of the guide field for a large distance from the magnets center. There is no "accidental" build up of single driving terms due to unfortunate choice of the phase advances per FODO cell. It is expected that a somewhat smoother curve of the guide field as a function of the distance from the center would improve the dynamic aperture. A detailed discussion and derivation of distortion functions is given in the appendix.

1. Introduction

The design of the superconducting dipole magnet is one of the crucial aspects of a large future hadron collider. The magnets costs - a significant part of the total costs of such a project /SSC84/- depend strongly on the magnet aperture and on the required accuracy of the magnet manufacturing and assembly.

The same parameters are very important for beam dynamics. The magnet aperture requirement is determined by the beam size at injection, the linear lattice design, and an operational need for free aperture to allow for injection errors and orbit distortions, essential for commissioning and optimizing the machine performance. The magnets imperfections are determining factors for the beam stability and the dynamic aperture.

An optimum magnet design implies that the physical aperture of the machine is nearly identical to the dynamic aperture. Otherwise, magnet aperture is wasted if it cannot be used by the beam. Magnet accuracy is a wasted effort if the dynamic aperture can not be used due to physical aperture restrictions.

In existing machines, the problem of matching the magnet design as well as possible to the beam dynamics requirements has been avoided by applying a certain safety factor. One wants to avoid these extra costs building a large future machine.

The field quality of a superconducting magnet is determined by persistent currents effects mainly at injection energy, the accuracy and mechanical stability of conductor placement, and by the design of the conductor arrangement. There is a distinction between designed and random field errors. We will concentrate on the systematic multipole errors, the sum of designed and average random errors in this report.

The aim of this study is to reveal how details of the systematic guide field errors are related to the dynamic aperture and the beam dynamics.

The reason why the TEVATRON has been chosen as a test lattice is quite obvious. The TEVATRON is the prototype of superconducting synchrotrons. In many aspects it is very similar to any future large machine.

The well tested tool for such investigations are tracking calculations using conventional kick codes. For the TEVATRON, such calculations were performed in the past /WIL83/, /GEL83/ and have been compared with the multipole structure of the TEVATRON dipoles. But it is very difficult to relate the results of tracking calculations with details of the magnet structure. In order to make sure not to be misled by accidental coincidences one has to perform a large number of tracking runs changing many parameters systematically. This is very costly ,time consuming, and after all, does not guarantee success.

Analytic methods are therefore a very desirable complement to tracking calculations. For the TEVATRON first attempts were made /WIL83/ using Moser transformations /MOS55/ to obtain the nonlinear distortions of phase space trajectories as a perturbation expansion. Recently, the lowest order contributions to these distortions have been introduced as 'distortion functions' /COL84/. We will use this expression in this report for nonlinear phase space distortions expanded to any order in perturbation theory.

The basic idea to obtain phase space distortions as a result of a canonical transformation is as follows:

The nonlinear dynamics is described by a nonlinear hamiltonian, which is a product of the nonlinear field coefficients and powers of the particle distance from the equilibrium orbit. The hamiltonian can be decomposed into fast oscillating (nonresonant) terms, constant (detuning) terms and slowly varying (resonant) terms. The constant and slowly varying terms dominate the particle motion. The fast oscillating terms are expected to cancel over many revolutions in the machine and can be treated as a distortion. Resonant terms, however, can be avoided by a careful choice of the linear machine tunes. If one finds a coordinate transformation into a new hamiltonian system where the new hamiltonian contains only constant terms for which the solution of the equations of motion is trivial, the whole nonlinear effect is described by the coordinate transformation back into the old system. The distortion functions used here are a perturbation expansion up to 2nd order of this transformation.

Though the concept of successive canonical transformations is well known and has been often described in the literature, explicit expressions for two degrees of freedom for any multipole order and for higher orders in the perturbation expansion are not easily found. Therefore details of the analytical model used here and the formulae on which the study is based are derived and presented in the appendices.

The numerical results presented in this report are obtained from the computer code CANOL /CAN85/ which calculates driving terms, distortion functions (up to second order perturbation theory incl. terms up to 24-pole) and resulting phase space trajectories.

The report is structured in the following way:

First the model which describes the TEVATRON will be presented. The analysis is based on this model.

In the following section application of the phase space distortion concept to the TEVATRON model will be presented and discussed.

Then numerical results are shown and the multipole errors and their impact on the dynamic aperture will be discussed order by order.

The appendices describe details of the formalism.

2. A Model for the TEVATRON

The analytic method is quite different from tracking calculations. In tracking one usually tries to describe the real lattice as closely as possible. Because of the complexity of the input it is very difficult to obtain a qualitative understanding of how the tracking results come about. Therefore the tracking code appears as a 'black box'.

An analytical method loses its advantage if one proceeds the same way. An important aspect of analytic calculations is that the formulation of the problem leads transparently to the final results. It is therefore essential to condense the complexity of a lattice into a model which contains only the essential features of the lattice.

In this sense a simplified model of the TEVATRON is introduced which is the basis of the study.

The TEVATRON consists of six sextants separated by straight sections. Each sextant arc is composed by 16 FODO cells. Four dipole magnets (l=6.127m, Θ=8.2 mr) are in each half cell. The arc is completed at the downstream end by 3 additional dipole magnets. The regularity of the arc is distorted by two missing dipoles in the 7th half cell from the upstream end of the arc.

In the model, nonlinear forces are concentrated in the middle of each half cell (or in the bending center of a group of 2 or 3 dipoles). Nonlinear forces in quadrupole magnets are neglected. Thus the straight sections enter in the description of the machine only by a betatron phase advance and different β functions in the first half cell and the last 3 dipoles.

There are two kinds of straight sections. This reduces the superperiodicity of the TEVATRON to 2: two high β straight sections with a maximum β of $\simeq$ 250m and four normal straights with $\beta \simeq 150$m. However the phase advance was designed to be the same for each straight section. Neglecting the difference in the β-functions over the first and last group of

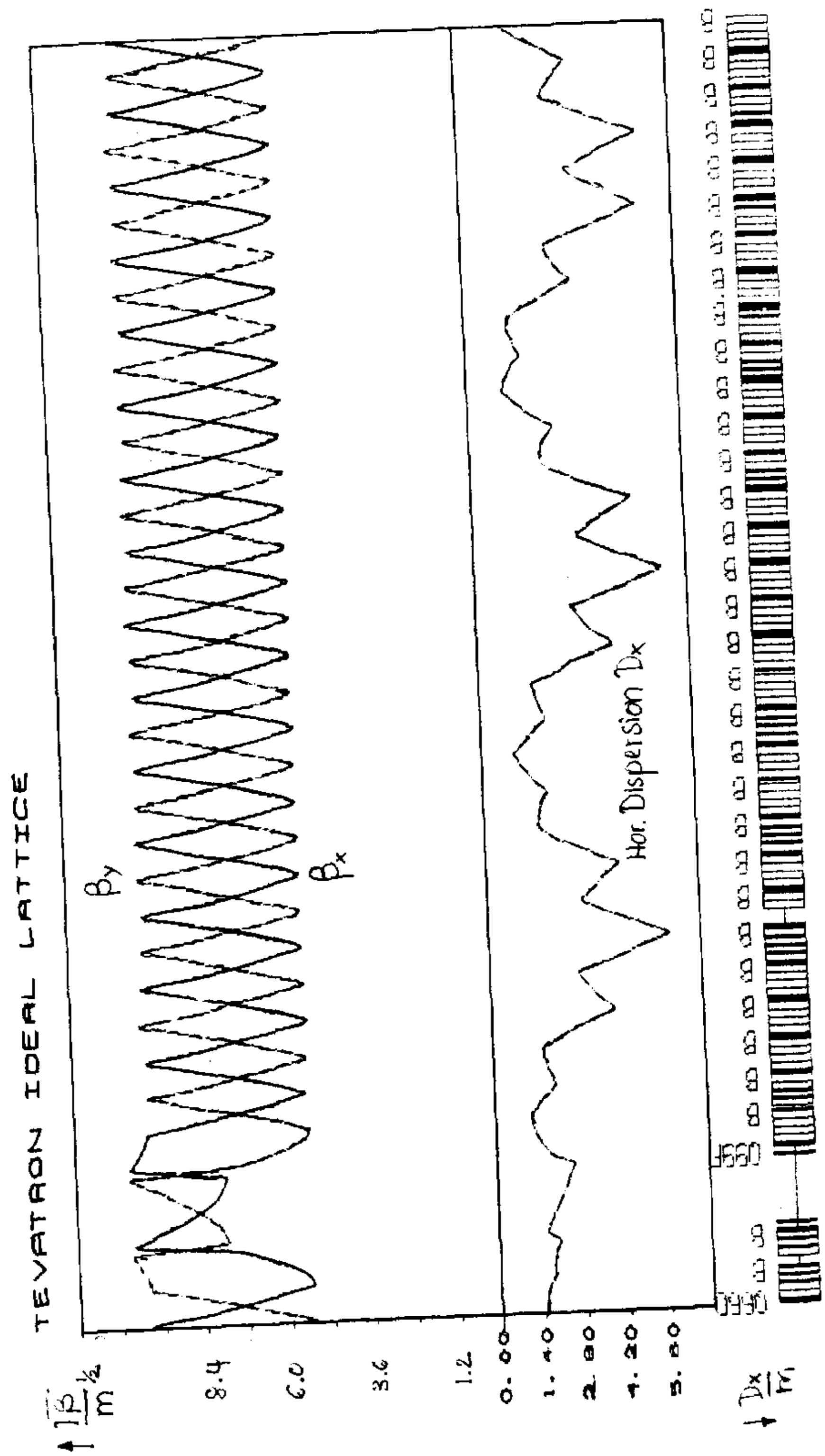


fig 2.1 TEVATRON Lattice Model with Linear Lattice Functions

dipoles in the arc for the high β and normal straight section, we have restored the sixfold symmetry. This is not only a large reduction of complexity without sacrificing much lattice information, but it also reduces the computing effort by a factor of 36.

The chromaticity correcting sextupoles are placed next to the quadrupole magnets in the real machine. In order to reduce the number of nonlinear kicks and the computing time, they have been moved to the center of the half cell in the model. The sextupole strengths have been scaled with

$$(\beta_x)^{3/2} \text{ for terms} \sim x^3 \quad \text{and with } \beta_x{}^{1/2} \cdot \beta_y \text{ for terms} \sim x \cdot y^2$$

which means the only error in doing so arises from neglecting the phase advance between the actual sextupole position and the middle of a half cell (≈17°). This is not a more severe approximation than concentrating the nonlinear kicks of the dipole magnets.

In the real machine some further minor distortions of supersymmetry are present which are neglected.

The multipole errors used in the model are the average of the measurements made for each individual TEVATRON dipole /HAN79/. The numbers are listed in table 2.1.

It should be mentioned at this point that it is not expected that the results of this study will quantitatively agree with measurements made at the real machine or with simulations based on the measurements of magnet errors. However it is expected that the qualitative results reflect the coherence of basic lattice parameters, magnet properties and beam dynamics.

The model lattice and basic linear lattice functions are shown in fig 2.1.

TABLE 2.1

Average Multipole Components Measured at $\frac{1}{4}$ inch relative field errors in units of 10^{-4}

	normal coefficients b_k	skew coefficients a_k
6-pole	0.99	0.38
8-pole	-.27	-.07
10-pole	-.76	-.07
12-pole	-.05	-.10
14-pole	6.69	0.15
16-pole	0.02	0.25
18-pole	-15.69	-.73
20-pole	0.01	0.42

3. Phase Space Distortions in the TEVATRON

The difficulties of using the concept of isolated resonance driving terms for a real machine like the TEVATRON are well known: Evaluation of driving terms for realistic cases very often results in a large number of equally important terms rather than one dominant one. Furthermore, a driving term is just one term in a fourier series of a component of the nonlinear field which dominates the rest of the series only if the distance to the resonance is close enough. This, however, is always avoided in real machine. Therefore driving terms or widths of isolated resonances calculated for a real machine are only a relative measure of the importance of a certain component of the nonlinear field.

Therefore it is more advantageous to use the phase space distortions as such a measure. First of all, they contain all harmonics of a certain component of the nonlinear force. If the total distortion is small, the lowest order distortion functions are a sufficiently accurate description of the nonlinear motion. Near the dynamic aperture the lowest order distortion function concept breaks down because distortions become very large and many higher orders in the perturbation expansion contribute. But even in this situation distortion functions are useful. The strongest terms of the distortions at the dynamic aperture are those terms which ought to be retained as dominant terms in the hamiltonian. An analysis of the phase space distortions therefore provides an excellent criterion for the selection of driving terms. Moreover the betatron amplitudes for which the distortion function concept obviously breaks down agree very well with the dynamic aperture obtained from the hamiltonian procedure having chosen the "right" driving terms. (This is not very surprising after a close look at the mathematics which determines the unstable fixed points and which determines on the other hand the amplitudes beyond which the distortion functions become unphysical (see below).)

If one is not interested in details of phase space trajectories but only in which are the dominant terms and in why are they dominant, one can do without the hamiltonian procedure and draw conclusions from the distortion functions alone.

In this sense we are calculating relative distortions $\delta\varepsilon$ for the betatron amplitudes (emittance or Lagrange invariant)

$$\varepsilon_x = x^2 \cdot \gamma + 2xx' \cdot \alpha + x'^2 \cdot \beta$$

of the form:

$$\delta\varepsilon/\varepsilon_x = 1 + \sum_{nm\nu\mu} \nu\sigma_{nm\nu\mu} J_x^{\frac{n-2}{2}} J_y^{\frac{m}{2}} \cos(\nu\Phi_x + \mu\Phi_y + \Phi_{nm\nu\mu}) \;/\; \sin\pi(\nu Q_x + \mu Q_y)$$

The distortion is calculated for a certain position in the lattice as a function of the betatron phase angle Φ. The variable J is the Poincare integral invariant $\int d\Phi \cdot \varepsilon(\Phi)/2\pi$ where ε is the distorted emittance. Note that the invariant phase space area would not change if the nonlinear forces were switched off adiabatically. It corresponds therefore to the radius of a circular linear phase space trajectory and will be referred to as the "undistorted" emittance or betatron amplitude. The integers n+m are the multipole order and $|\nu|+|\mu|$ is the order of the nonlinear resonance potentially driven by the component.

A similar formula is given for the distortion in the y-y'-plane.

To obtain the distortion of the betatron phase $\delta\Phi$ as a function of the undistorted amplitude J one has to invert the following expression:

$$\delta\Phi_x = 1 + \sum_{nm\nu\mu} \frac{n}{2}\sigma_{nm\nu\mu} J_x^{\frac{n-2}{2}} J_y^{\frac{m}{2}} \sin(\nu(\Psi_x-\delta\Phi_x)+\mu(\Psi_y-\delta\Phi_y)+\Phi_{nm\nu\mu})/\sin\pi(\nu Q_x+\mu Q_y)$$

where Ψ is the undisturbed phase. This form of the distortion is the same for all orders of the perturbation expansion. The coefficients σ depend on the linear lattice functions and the multipole coefficients and are given in appendix A.(sections A7,8,9)

The phase space distortions have been calculated for the test lattice presented in the previous section. The tunes have been chosen carefully in order to avoid resonance enhancement of the distortions.

Fig 3.1 shows a projection of a distorted phase space trajectory on the x-x' and y-y' plane for different emittances. The undistorted emittances have been chosen to be equal for x and y (round beam). Because the phase phase trajectories in x-x' depend on the phase angle in y and vice versa, points with the same phase angle in y and x respectively have been chosen. The projection can therefore be considered as a cut through the 4 dimensional phase space for (approximately) constant vertical betatron phase.

The distortions include first order effects up to 20-pole and second order effects up to the order n+m=10 (that includes interference of the strong 18-pole and 6-pole, 16-pole and 8-pole, 14-pole and 10-pole, 12-pole and 12-pole). The outermost trajectory is the trajectory for which the slope of the distorted amplitude $\delta\varepsilon$ as a function of the undistorted amplitude J is zero. This is expected to be a trajectory very close to the dynamic aperture.

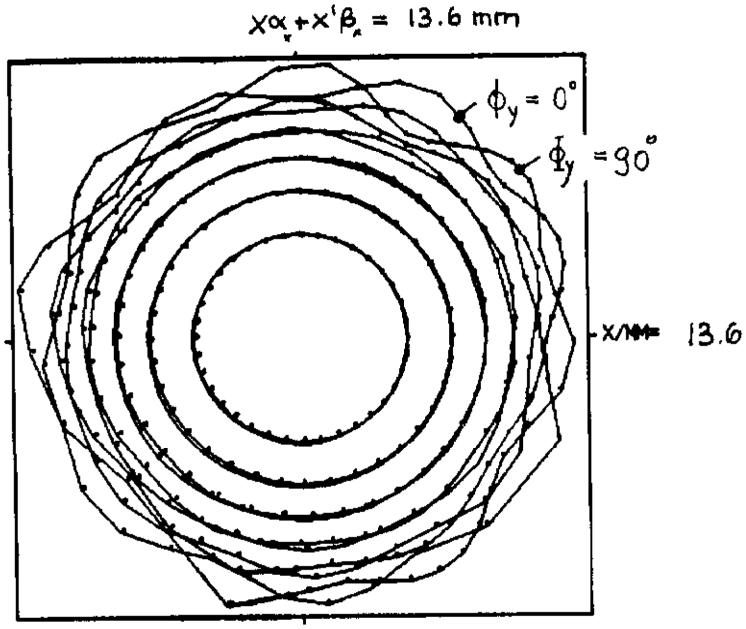


fig 3.1.a
x-x'

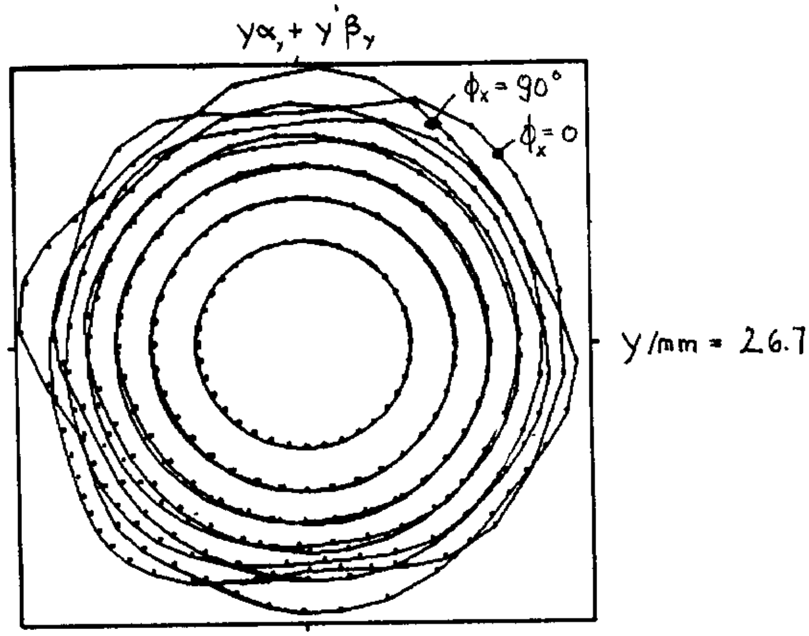


fig 3.1.b
y-y'

fig 3.1 Projections of a phase space trajectories
a) on x-x' plane for $\Phi_y \simeq$ constant
b) on y-y' plane for $\Phi_x \simeq$ constant

In fig 3.2 the minimum value of the distorted emittance $\varepsilon(\Phi)$ is shown as a function of the undistorted emittance J. The dynamic aperture is expected to occur when the slope of this curve is zero (dashed line). Comparison with tracking calculations (dotted line) using the RACETRACK kick code /WRU84/ shows good agreement with the distortion function result. The dash-dotted curve has only first order terms which shows that the contribution from the second order terms are approximately 1/4 of the first order terms. This shows the importance of higher order contributions near the dynamic aperture.

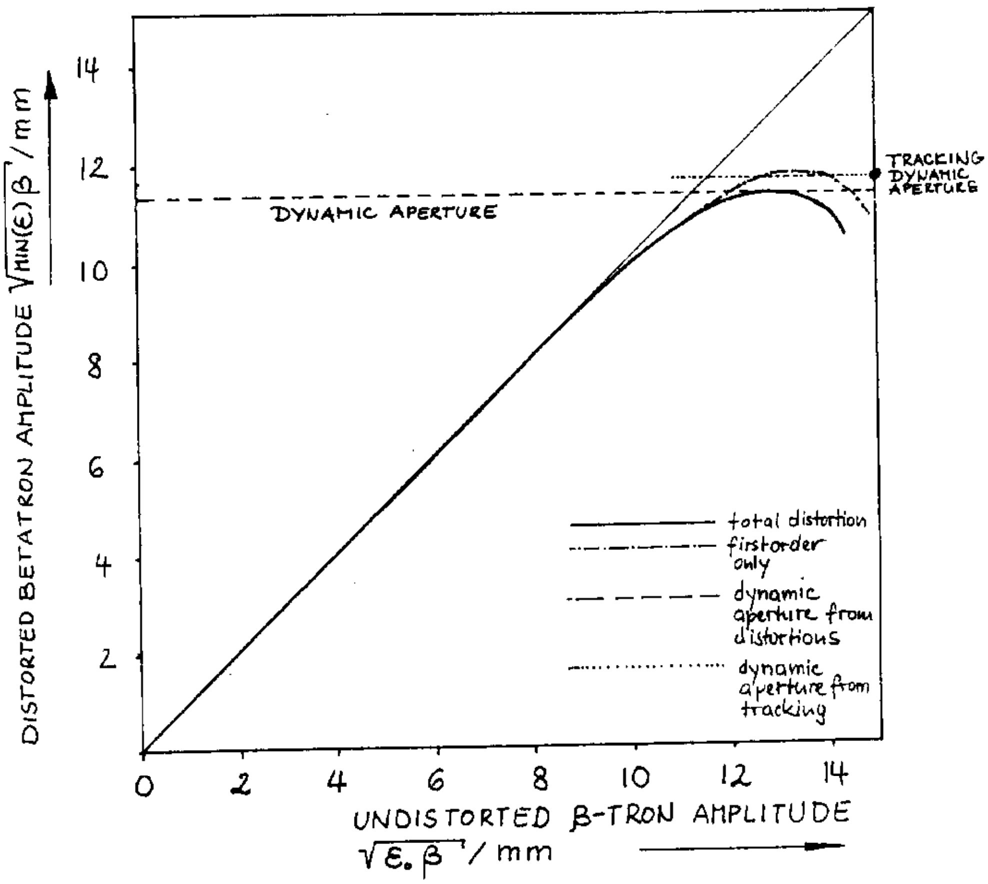


fig 3.2 Dynamic Aperture Derived from Distortion Functions
Minimum of Distorted Horizontal Emittance ε as a
Function of the Undistorted Emittance J
Solid Curve : Total Distorted Emittance
Dash-Dotted Curve: First order Terms Only

The nonlinear tune shift as a function of the undistorted emittances J_x+J_y is shown in fig 3.3a and 3.3b. The tune shift is rather linear in the range between 0 and 3 π mm mr and it becomes strongly nonlinear near the dynamic aperture.

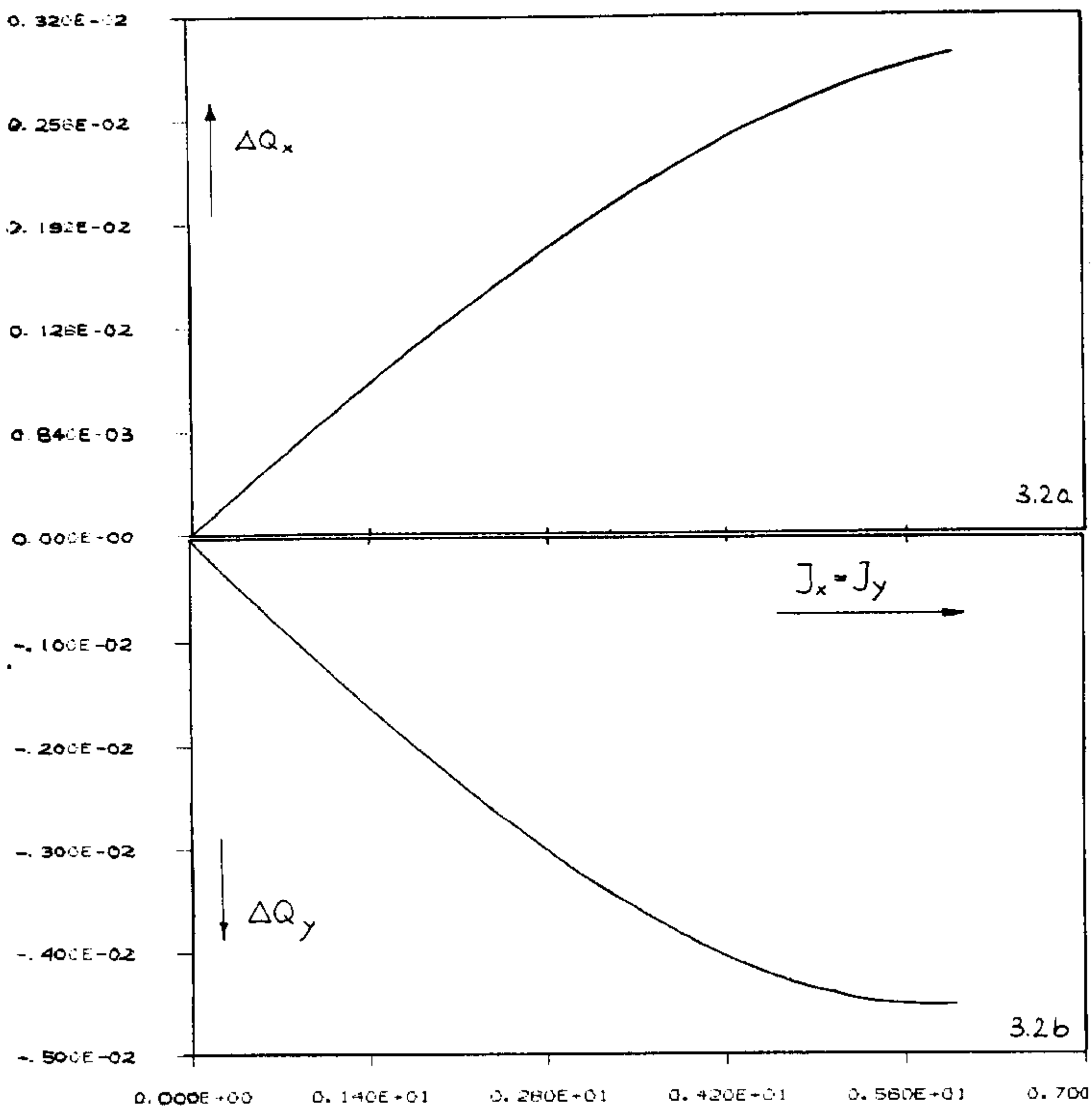


fig 3.3 a Horizontal Tune Change with Undistorted Emittance J_x+J_y

fig 3.3 b Vertical Tune Change with Undistorted Emittance J_x+J_y

4. Detailed Discussion of Multipole Errors Order by Order

In order to understand the results presented in the preceding section, we have to decompose the total distortion into contributions originating from the different multipole components of the nonlinear field.

A look at the resonance denominator spectrum (fig 4.1) confirms that there are only few terms among the distortion functions which are enhanced because they are close to a resonance. This happens for the terms with $7Q_x-2Q_y$ for which one obtains a resonance enhancement of 54. We will see later that the dynamic aperture is not very much affected by these terms. Besides this the spectrum looks very well balanced with most of the terms near unity.

We first compare the spectrum of phase space distortions for an emittance which corresponds to the beam size at high energy (ε=0.2 π mm mr) and the emittance near the dynamic aperture (ε=6 π mm mr). (The distortion amplitudes are given by eq. 9.7 in appendix A for all terms characterized by $n+m\le12$ and $|\nu|+|\mu|\le12$ which includes 915 terms.) As one expects from the above form of the distortion, for the small amplitudes (fig 4.2 a,b,c) the low order multipoles (6-poles, 8-poles) dominate the distortions which are confined to values below 0.2%. Near the acceptance limit (fig4.3a,b,c) only 14,18 and 20-pole are important. The distortion amplitudes reach 25%.

The most important contributions (at least up to an order $n+m=10$) are first order terms (fig's 4.2a, 4.3a) for small and large amplitudes as well. However the second order terms which consist of x-like terms and y-like terms (see appendix A) are an important contribution at the large amplitude and cannot be neglected. This reflects the fact that the perturbation expansion diverges near the dynamic aperture and many higher orders contribute unless there is a dominating lower order term enhanced by a small resonance denominator which we can exclude in our case. Fig 4.2b,c and fig 4.3b,c show that the most important 2nd order contributions are terms with $n+m=10$. They are produced by the interference of the strong 18-pole and the 6-pole and the interference of the 14-pole with the 10-pole.

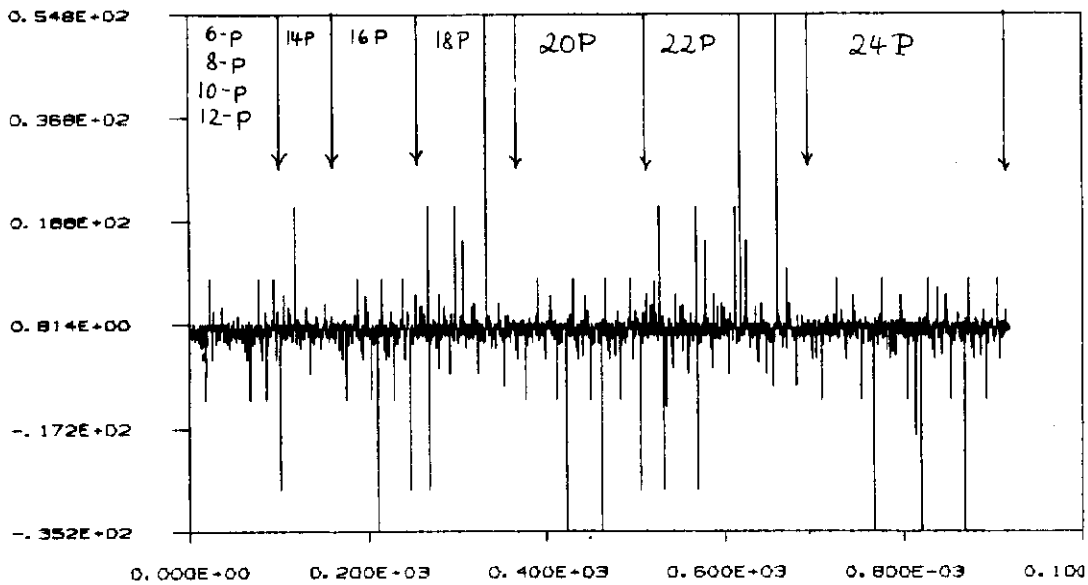


fig 4.1 Resonance denominator spectrum $1/\sin\pi(\nu Q_x+\mu Q_y)$ the absissa is the numbering of the terms. The terms originating from the same multipoles are grouped together.

fig 4.2a Spectrum of Phase Space Distortions for $J_x=J_y=.2\pi$mm mr. First Order Pertubation Theory, same absissa as 4.1

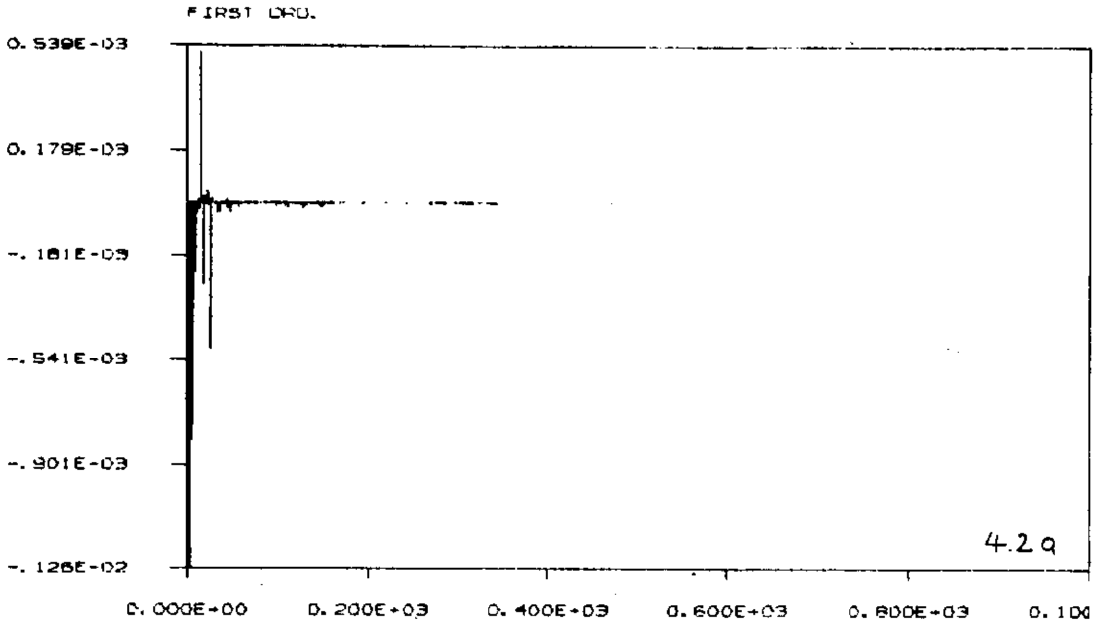
fig 4.2b Spectrum of Phase Space Distortions for $J_x=J_y=0.2$mm mr. 2nd Order Pertubation Theory, x-like terms

fig 4.2c Spectrum of Phase Space Distortions for $J_x=J_y=0.2$mm mr. 2nd Order Pertubation Theory, y-like terms

fig 4.3a Spectrum of Phase Space Distortions for $J_x=J_y=6\pi$mm mr. First Order Perturbation Theory, same absissa as 4.1

fig 4.3a Spectrum of Phase Space Distortions for $J_x=J_y=6\pi$mm mr. 2nd Order Perturbation Theory, x-like terms

fig 4.3a Spectrum of Phase Space Distortions for $J_x=J_y=6\pi$mm mr. 2nd Order Perturbation Theory, y-like terms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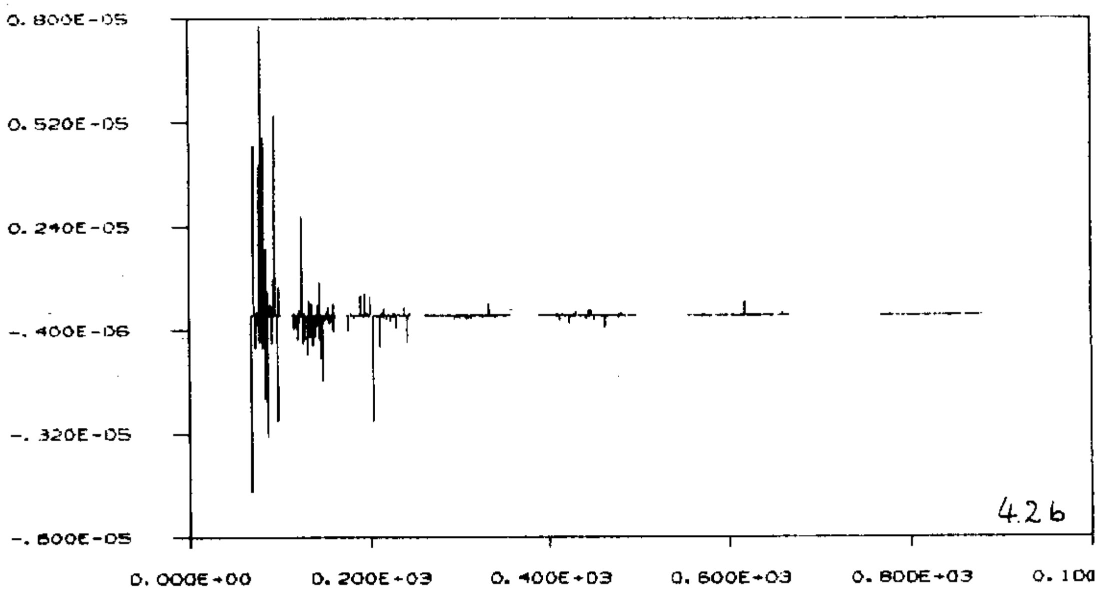
4.2a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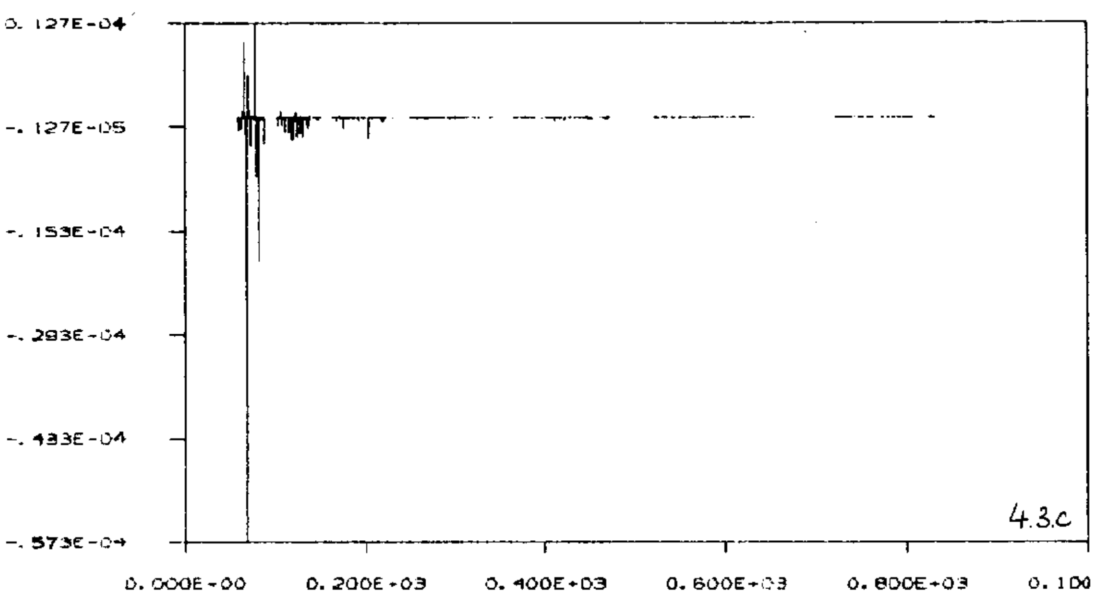
0.100
4.2b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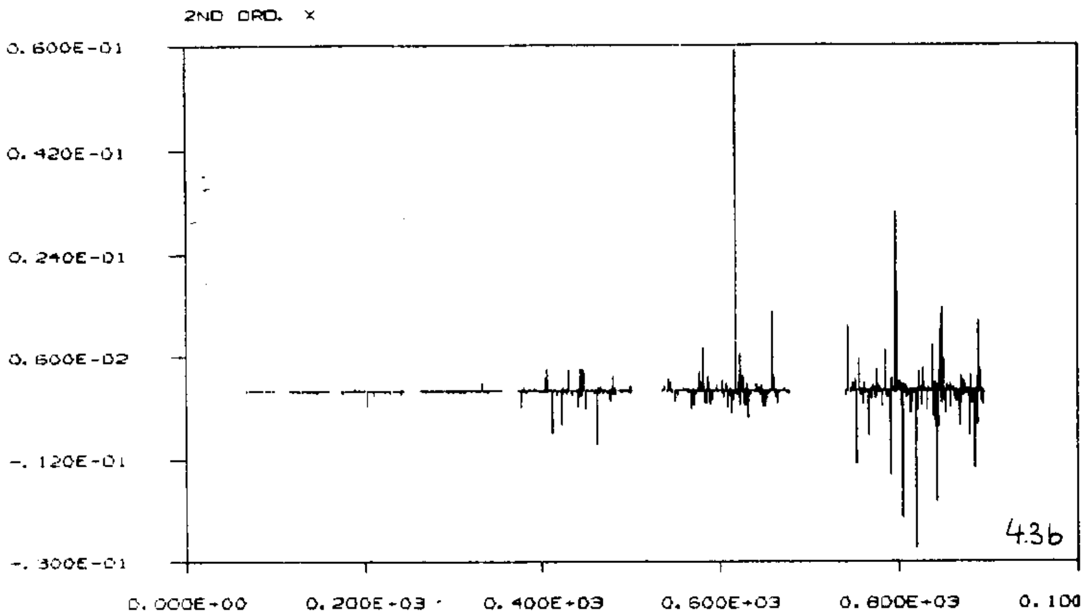
4.3.c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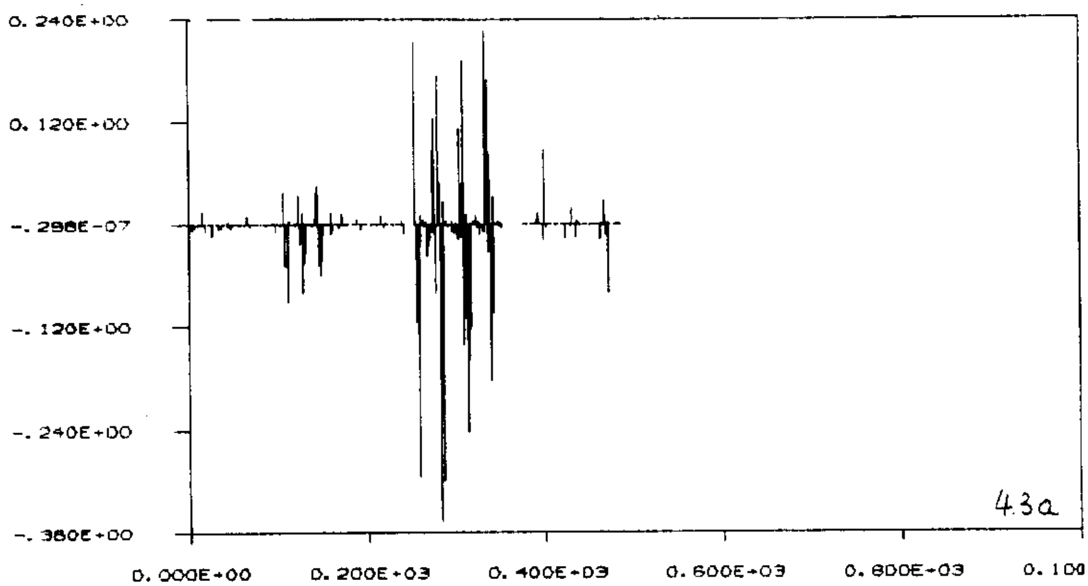
4.3b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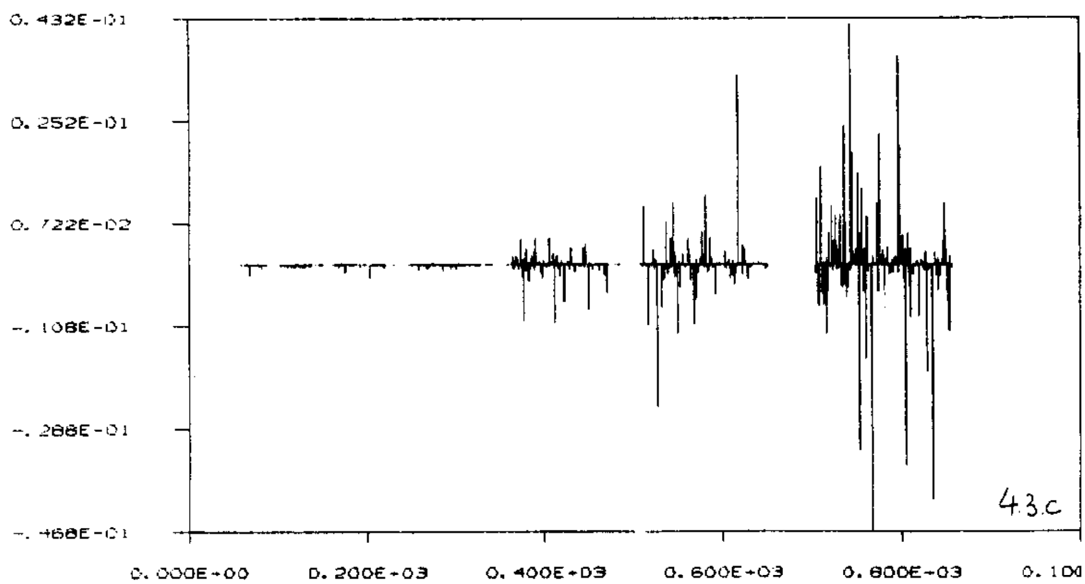
0.100
4.3a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0.800E+03
0.100
4.3c

It is also very instructive to compare the strongest contributions to the distortion from the different multipole components and plot the appropriate distortion as a function of the undistorted emittance ($J_x=J_y$, see fig 4.3a,b,c). At emittances below 1 π mm mr the sextupole contributions are by at least 1 order of magnitude larger than any other component. At J=1.5 πmm mr however the situation has changed. Above this amplitude 14-pole and 18-pole components are by far the strongest contributions and at the dynamic aperture J=5.4πmm mr, the 18-pole component is almost an order of magnitude larger than any other multipole term. The 12-pole and the 16-pole and normal 20 pole are the least important contributions whereas the octupole is comparable with the sextupole (fig 4.4a).

The skew terms (fig 4.4b) are in general smaller than the normal terms. This is simply because the skew components of the field are smaller than the normal components. An exception is the skew 20-pole which becomes almost as strong as the normal 14-pole at the dynamic aperture.

TABLE II

Strongest Distortion Contribution from Each Multipole, J=6πmm mr

1st order normal multipole	n	m	ν	μ	rel.dist.	1st order skew multipole	n	m	ν	μ	rel.dist.
6	1	2	1	2	.007	6	2	1	2	-1	.004
8	2	2	2	2	.008	8	3	1	3	1	.001
10	3	2	1	2	.006	10	4	1	2	-1	.001
12	4	2	2	2	.003	12	5	1	3	1	.002
14	1	6	1	-4	.091	14	2	5	0	3	.002
16	2	6	2	2	.002	16	5	3	3	1	.011
18	3	6	1	2	.348	18	2	7	0	7	.025
20	small					20	7	3	1	-3	.035

2nd order normal terms	n	m	ν	μ	rel.dist.	2nd order skew terms	n	m	ν	μ	rel.dist.
18	5	2	7	-2	.002						
20	4	4	2	2	.010	20	5	3	5	3	.006
22	5	4	7	-2	.060	22	2	7	2	-5	.025
24	4	6	4	4	.043	24	5	5	5	3	.047

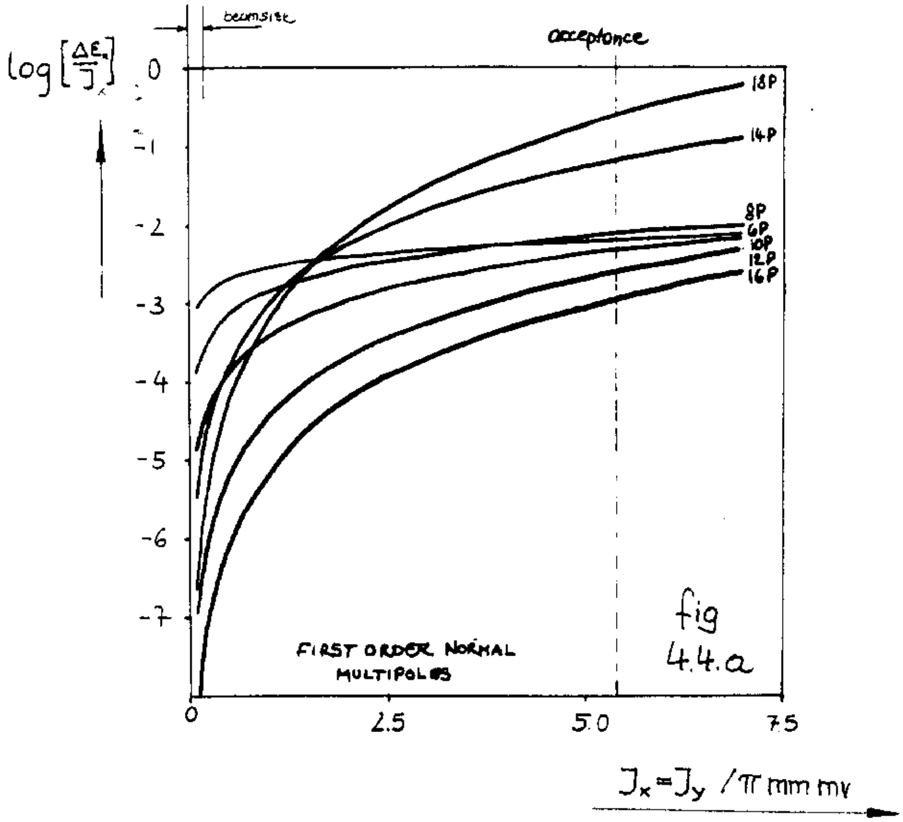

fig 4.4.a

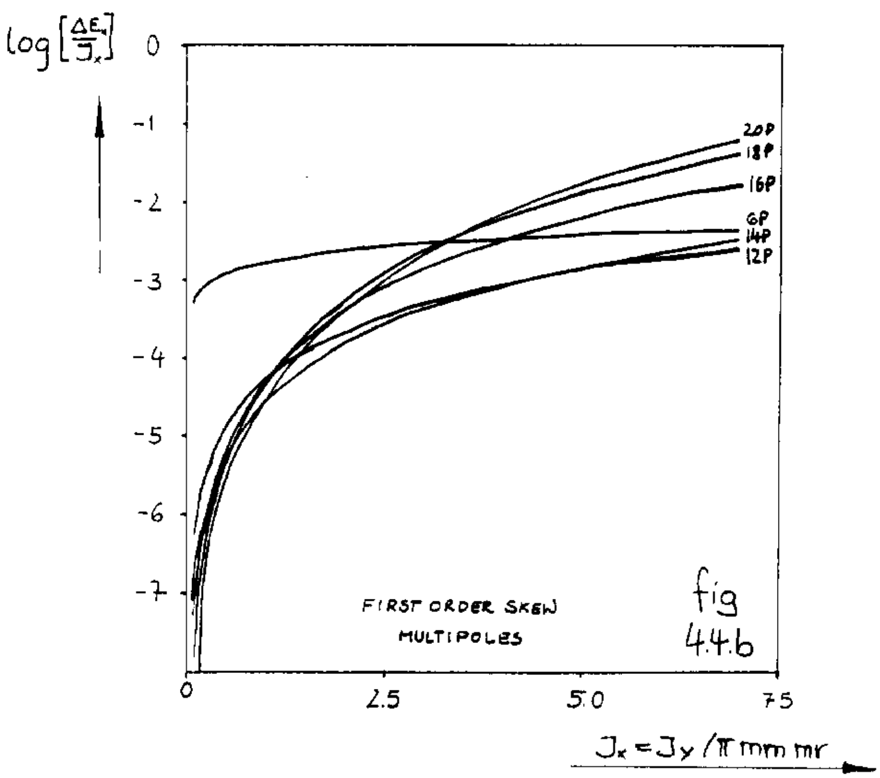

fig 4.4.b

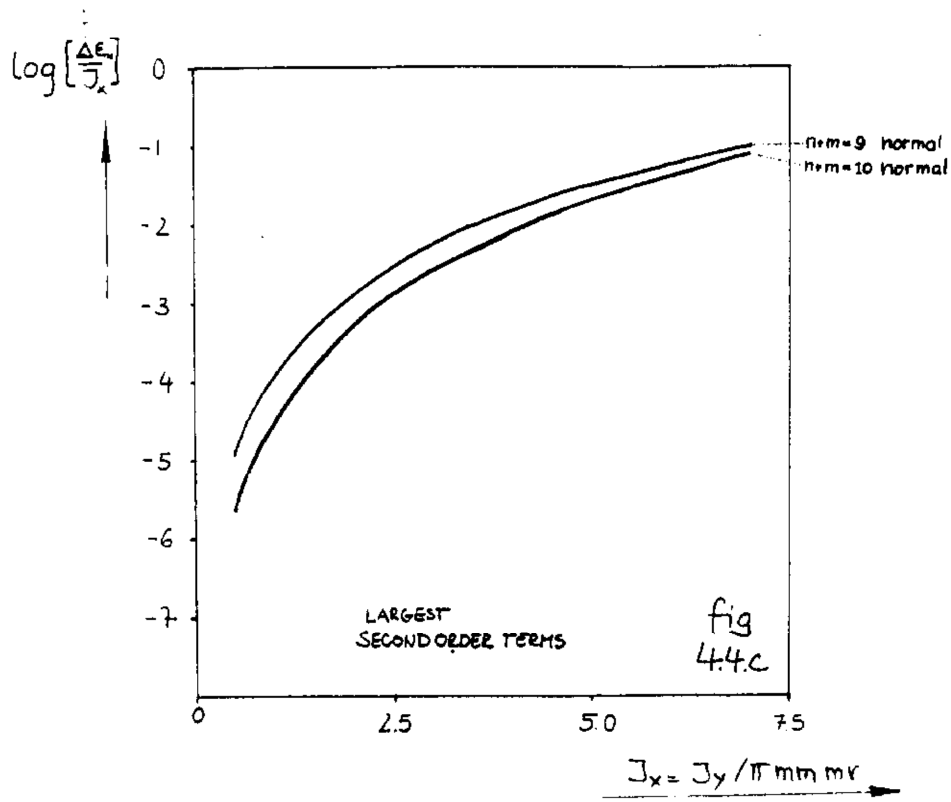

fig 4.4a Strongest <u>First Order</u> Contribution to the relative distortion as a function of the emittance $J_x = J_y$ from <u>normal</u> multipoles

fig 4.4b Strongest <u>First Order</u> Contribution to the relative distortion as a function of the emittance $J_x = J_y$ from <u>skew</u> multipoles

fig 4.4c Strongest <u>2nd Order</u> Contribution to the relative distortion as a function of the emittance $J_x = J_y$ from <u>normal</u> multipoles

Table II gives a listing of the strongest contributions to the distortion from each multipole component and its characteristics.

There are several factors which cause a particular term to be important or dominating:

* The multipole component which drives the term is large.

* The resonance denominator is small.

* The contributions from the different nonlinear elements around the ring build up rather than cancel each other.

* The term is a coupling term with n close to m and drives a low order resonance ($|\nu|+|\mu|$ smaller n+m, large binominal factors).

* The distortion phase of that term has to be such that there is a positive interference with other strong terms.

An important aspect of the design of the magnet and the choice of phase advances and tunes should be to avoid the coincidence of all these factors which can result in an accidental dominance of a few terms which can cause a drastic reduction of the dynamic aperture.

We want to analyze the strongest contribution to the distortion under these conditions.

The 18-pole component together with the also strong 14-pole describes the break off of the guide field. It is not very surprising that this multipole has the largest impact on the dynamic aperture

The resonance denominators of almost all the strong terms considered so far are not particularly small with the exception of the terms with $\nu=7$, $\mu=-2$ where the enhancement is 54. The strongest 18-pole term ($n=3, m=6, \nu=1, \mu=2$) is only enhanced by a factor of 1.7 which means that the dynamic aperture is not reduced by an unfortunate choice of the tunes.

It is furthermore not surprising that the terms with a large binominal factor

$$\frac{(n+m-1)!}{(\frac{n+\nu}{2})!\ (\frac{n-\nu}{2})!\ (\frac{m+\mu}{2})!\ (\frac{m-\mu}{2})!} \qquad \text{(see appendices)}$$

are large for the terms which cause a large distortion.

Finally we have to consider the build up of the terms as a superposition of the nonlinear elements in the ring. If one neglects the effect of the missing magnets in the structure, one can use the formula A13.3 which gives the distortion amplitude for a regular FODO structure as a function of the phase advance per cell and the number of cells. Assuming $\Phi_x \simeq \Phi_y$ one finds for all strong terms a build up factor smaller than one.

Fig. 4.5 shows the build up factor

$$\sin(\tfrac{k}{2}(\nu+\mu)\Phi_c)/\sin((\nu+\mu)\Phi_c/4),\ k=16,\ \Phi_c \simeq 68°,\ \nu+\mu = 1,\dots 12$$

for several phase advances near 68° as a function of the phase multiplier ($\nu+\mu$). The TEVATRON phase advance with 68.8° is fairly well chosen. The build up of terms could be improved however by lowering the phase advance to 67° which corresponds to a machine tune of 18.9 instead of 19.4 (neglecting the missing magnets which maximally add 0.75 to the built up factor).

A list of the strongest 18-pole contributions is given in table III.

TABLE III

Characteristics of strong 18-pole distortions

term n m ν μ	binominal factor	resonance enhancem.	build up factor	relative distortion $\delta\varepsilon/\varepsilon$ for $J=6\pi$mm mr
5 4 3 0	420	1.18	0.66	0.141
5 4 3 -2	280	3.15	0.61	0.100
5 4 1 2	560	1.70	0.66	0.243
5 4 1 0	840	1.54	0.61	0.157
5 4 1 -4	114	4.21	0.66	0.121
3 6 1 2	420	1.70	0.66	0.348
3 6 1 0	560	1.54	0.61	0.127
3 6 1 -4	168	4.21	0.66	0.301
3 6 3 2	140	1.41	0.79	0.126
3 6 1 6	28	6.37	1.11	0.175
7 2 7 -2	4	54.42	0.79	0.227
7 2 5 0	56	2.54	0.79	0.171
7 2 3 0	198	1.19	0.66	0.144
7 2 1 0	280	1.54	0.61	0.184

The total distortion as shown in figs 3.1 and 3.2 is essentially the superposition of these terms with proper distortion phases. There are 15 singular distortion contributions larger than 10% which result in a total distortion of 16%. This shows that the distortion phases are very well distributed for our test lattice.

The second order terms (fig 4.4c) have to be considered at emittances larger than 2.5 πmm mr and compete with the normal 14-pole and the skew 20 pole at the dynamic aperture.

The strongest contribution (6%) from the second order perturbation theory to the phase space distortions is derived from the term $n=7, m=4, \nu=7, \mu=-2$. It is the only important term which is enhanced by a small denominator by a factor of 54. It is the result of interference between mainly first order 14-pole and 8-pole terms. Without enhancement these terms cause distortions smaller than 1%.

Besides this single 14-pole - 8- pole interference, the most important second order distortions (2%-5%) come from 18-pole - 6-pole interference terms. The strongest (4.7%) is the term $n=4, m=6, \nu=4, \mu=4$. It is enhanced by a factor of 6.09. Interference between 18-pole and 6-pole results in about ten times larger phase space distortions than the interference of 14-pole and 8-pole.

There are 16 combinations of 1st order sextupole and first order 18-pole contributing to $n=4, m=6, \nu=4, \mu=4$. It is not very surprising to find the strongest 18-pole terms among these contributions. The build up of the strongest pair of lrst order terms ($n=1, m=2, \nu=1, \mu=2$ + $n=3, m=6, \nu=3, \mu=4$) as a result of a double sum over the lattice elements (eq.A9.3) is not particularly strong as one verifies quickly by checking the denominators in eq A13.5 (which is the evaluation of the double sum for a regular lattice).

We can conclude this section by stating that there are no important accidental enhancements of particular terms contributing to the distortion function due to the choice of tunes or due to unfortunate lattice design. Thus there is no accidental reduction of the dynamic aperture in the TEVATRON.

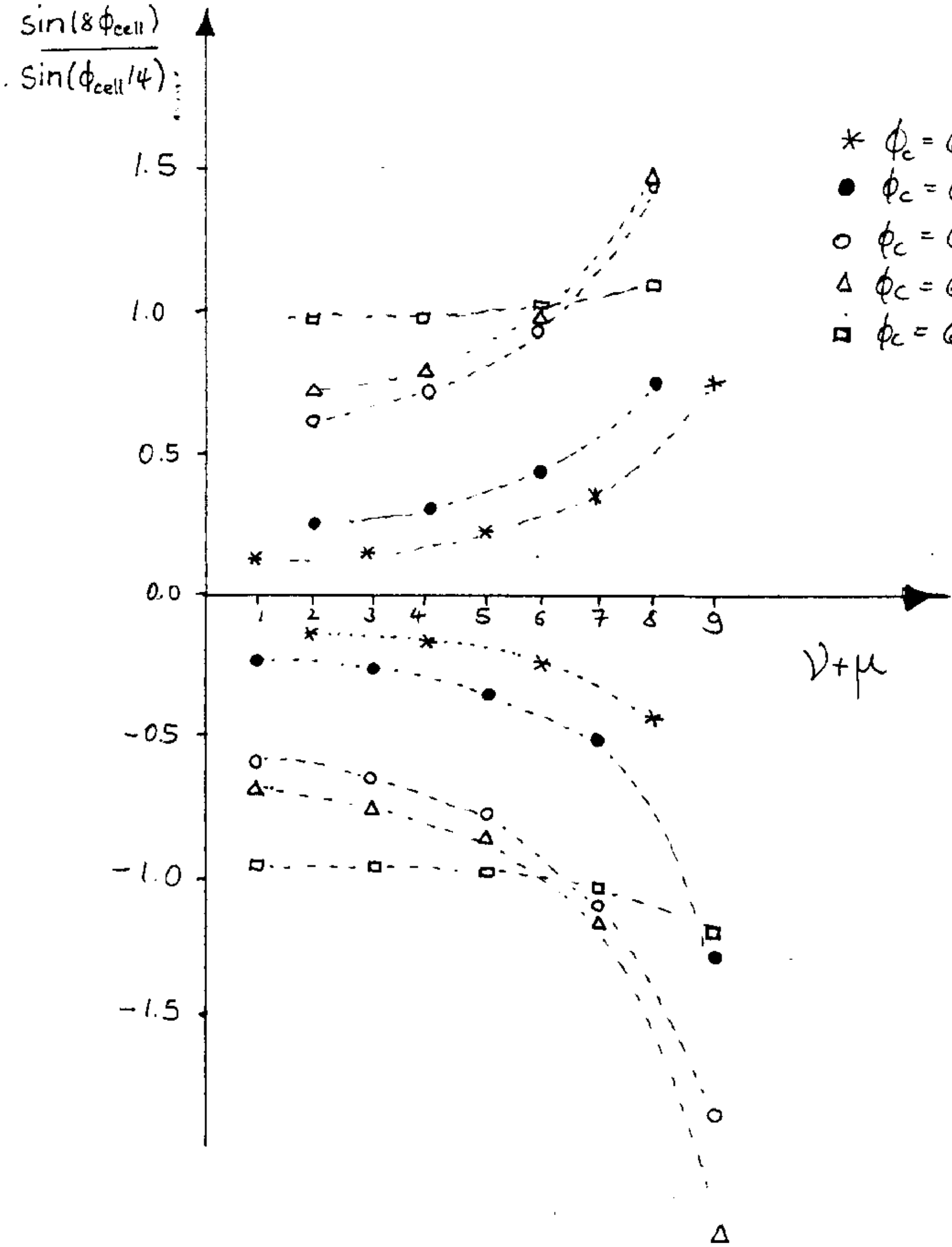

fig 4.5 Build up of Distortion Functions over the Lattice for Different Phase Advances/FODO cell

5. Conclusions

The discussion in the previous sections leads to the conclusion that the TEVATRON dynamic aperture is essentially given by the break off of the magnetic guide field. There are apparently no features of the magnet multipole structure which are enhanced by the beam dynamics and cause surprisingly large effects on the dynamic aperture. Moreover, the multipole structure of the magnet is very well reflected by the spectrum of phase space distortions which are closely related to the dynamic aperture. The strongest phase space distortions at the dynamic aperture are produced by the strong 18-pole. That are the multipole components which describe the break-down of the guide field. Interference effects of the strongest multipole components among each other are important for the dynamic aperture but, at least up to 24-pole effects, are not dominating.

The characteristics of the distortion spectrum suggest a slightly different multipole structure. Because the 18-pole is much stronger than 10,12,16 and 20 pole one expects that a somewhat smoother break off of the guide field emphasizing a little bit more those components. Reducing the 18 and 14 pole leads to a larger dynamic aperture and a more effective use of the available physical aperture. This hypothesis will have to be analyzed on the basis of magnet design and field calculations.

The analysis in the previous section is by far incomplete and is intended to be a first step. At this stage we are not allowed to extend of these qualitative results beyond the machine model used for the calculations. The conclusions may even change qualitatively for a different lattice design. Thus we cannot not derive yet a general rule which applies to all machines and each magnet design.

It is also clear that as a complement to investigation of systematic multipole errors it is also necessary to analyze the impact of random multipole errors.

One major goal of this study was to demonstrate how analytical methods can be used to understand tracking results.

A large amount of future analytic and complementary tracking calculations will be necessary to provide the magnet builders with a beam dynamics criterion for an optimum magnet design.

Acknowledgement

This work was done during my stay at FERMILAB from June 1983 until July 1984 and between February 1985 and July 1985. I am very grateful to all my FERMILAB colleagues from the accelerator division for many discussions. My special thanks go to Don Edwards,Rol Johnson and co-workers. I am also obliged to my DESY colleagues J.Rossbach and R. Brinkmann for clarifying discussions.

APPENDIX A

Expansion of Phase Space Distortions and the Slowly Varying Hamiltonian

1. Introduction

In the following sections, phase space distortions ('distortion functions') and the slowly varying hamiltonian will be expanded in a perturbation series. The results are expressed in multipole expansion coefficients and linear lattice functions.

The traditional procedure using a generating function mixed in a new and an old set of canonical variables as introduced by Moser/MOS55/ and applied to accelerator problems by Schoch /SCH57/ and Hagedorn /HAG57/ is followed.

In the past, the examination of the slowly varying hamiltonian has been emphasized. It has been attempted to parameterize the beam dynamics by the strength of isolated resonances. Much effort has been spent to define and to study the width of nonlinear resonances /GUI71,73/.

In a real accelerator or storage ring however, one tries to avoid situations where just one or a few terms of the hamiltonian are important. This is accomplished by a careful magnet design and the appropriate choice of the working point.

Therefore in practice, one usually finds many equally important components in the hamiltonian rather than one strong term and the model of a single isolated resonances fails to describe the beam dynamics.

In such cases, the dynamics may be characterized much better by a transformation function of the canonical variables into a new system where the hamiltonian is trivial. Tom Collins called this transformation function 'Distortion Functions'/COL84/. Contrary to the slowly varying hamiltonian, distortion functions contain all harmonics of the nonlinear field distribution around the machine.

An important property of the distortion functions is that they are given in an expansion in the nonlinear field strength and the particle's transverse oscillation amplitude. It is well known that the expansion converges only as long as the total nonlinear effect is small. Near the dynamic aperture, where the nonlinear effects become dominant, the concept of distortion functions has to be used with great care.

Besides the traditional method described here, more recently Lie algebraic methods have been used to derive distortion functions /DEB69/. First applications to accelerator problems have been made /MIC85/ which look very promising.

2. Hamiltonian Formulation of Particle Motion with Nonlinear Fields

We start with a linear machine with no distortions and no linear coupling. The only forces acting on the particles are linear restoring forces due to normal magnetic quadrupole and dipole fields. The particle dynamics is derived from a linear hamiltonian G:

$$G = \frac{1}{2} x'^2 + \frac{1}{2} y'^2 + k_x(s)\, x^2 - k_y(s)\, y^2 \tag{2.1}$$

Here, x and y are the particle transverse positions with respect to the closed orbit; x' and y' are the slopes of the trajectories which are the canonical momenta if no longitudinal magnetic fields are present. The independent variable is the longitudinal position on the closed orbit s. The linear restoring forces are represented by functions $k_{x,y}(s)$. The solutions of the equations of motion

$$\partial G/\partial x = -\partial x'/\partial s \;;\quad \partial G/x' = \partial x/\partial s \;;\; x'=\partial x/\partial s \tag{2.2}$$

for x and y as a function of s are given in terms of the linear lattice functions $\beta(s)$ and $\alpha(s)$ and the phase advances $\Phi(s)$ for x and y plane respectively.

$$x = \sqrt{2\varepsilon_x\beta_x(s)}\,\cos\,(\Phi_x(s) + \Phi_x); \qquad y = \sqrt{2\varepsilon_y\beta_y(s)}\,\cos(\Phi_y(s)+\Phi_y) \tag{2.3}$$

$\varepsilon_{x,y}$ and $\Phi_{x,y}$ are constants of motions.

Sources of nonlinear forces are e.g. sextupole fields for chromaticity compensation and field imperfections of quadrupole and dipole magnets. Such nonlinearities contribute to the hamiltonian by the longitudinal component of the vector potential of the nonlinear magnetic fields which is expressed in a multipole expansion in the transverse particle coordinates x,y with respect to the middle of the nonlinear element:

$$G = \frac{1}{2} x'^2 + \frac{1}{2} y'^2 + k_x(s)x^2 + k_y(s)y^2 + \sum_{nm} a_{nm}(s)\, x^n y^m \tag{2.4}$$

Because the magnetic field has to satisfy Maxwells equations, the multipole coefficients a_{nm} are related by:

$$\nabla B = 0 \Rightarrow a_{n+2,m} + a_{n,m+2} = 0 \qquad (2.5)$$

(for the relationship of the a_{nm} with the familiar coefficients a_n and b_n see appendix B)

If the nonlinear fields are small distortions of the linear restoring forces, it is desirable to keep the concept of linear lattice functions. In order to express the solutions x(s),y(s) for the distorted hamiltonian in terms of the linear lattice functions, the 'linear' constants of motion ε and Φ must vary (variation of constants). If one inserts the solutions for x and y with varying constants in the equation of motion, one obtains a system of differential equations for ε and Φ which is of hamiltonian form where Φ play the role of a generalized coordinate and ε the role of the canonically conjugate momentum. The hamiltonian for this system contains the nonlinear distortions only. The transformation to the new canonical variables ε and Φ is a standard procedure in classical mechanics (transformation to action and angle variables).

$$H = \sum_{nm} a_{nm}(s) x^n y^m; \qquad \frac{\partial \varepsilon_x}{\partial s} = -\frac{\partial H(\varepsilon_x,\varepsilon_y,\Phi_x,\Phi_y)}{\partial \Phi_x} \; ; \quad \frac{\partial \Phi_x}{\partial s} = \frac{\partial H(\varepsilon_x,\varepsilon_y,\Phi_x,\Phi_y)}{\partial \varepsilon_x} \qquad (2.6)$$

The hamiltonian has to be expressed by the new canonical variables ε and Φ. It is convenient to change the independent variable from s to the machine azimuth Θ. The hamiltonian then has to be multiplied with the scale factor between both variables:

$$R = \int ds/2\pi.$$

Expressing the cosine-function in exponential form, one obtains:

$$H = \bar{R} \sum_{nm\nu\mu} \binom{n}{\frac{n-\nu}{2}} \binom{m}{\frac{m-\mu}{2}} a_{nm}(\Theta) \left(\frac{\beta_x}{2}\right)^{\frac{n}{2}} \left(\frac{\beta_y}{2}\right)^{\frac{m}{2}} \varepsilon^{\frac{n}{2}} \varepsilon^{\frac{m}{2}} e^{i\left(\nu(\Phi_x(\Theta)+\Phi_x)+\mu(\Phi_y(\Theta)+\Phi_y)\right)} \qquad (2.7)$$

The ν and μ are integers with $\nu \in \{-n,-n+2,\ldots,n-2,n\}$
and $\mu \in \{-m,-m+2,\ldots,m-2,m\}$

3. New Hamiltonian Containing only Slowly Varying Terms

It is well known /LIA66/ that, in general a nonlinear system is nonintegrable and solutions expressed by invariants and periodic lattice functions as in the linear case don't exist.

Solutions of the problem have always to be restricted to two extreme cases:

a) the total impact of the nonlinear fields is small or

b) only one or a few components of the nonlinear hamiltonian dominates the motion.

The aim of the expansion below is to advance as far as possible from these extreme cases in the region of interest for accelerators and storage rings.

The advantage of the above formulation of the dynamical system is that it allows one to extract from the complicated hamiltonian those terms which are important for the particle motion while the rest is treated in perturbation expansion.

We will try to find another set of canonical variables belonging to a new hamiltonian which contains only those 'important' terms. If the variation of the hamiltonian terms with the independent variable Θ is fast compared with the machine period, the effect of such terms is expected to cancel over many periods of the particle motion. Only the parts of the hamiltonian which vary slowly are expected to be important.

Before we proceed further, we want to factorize the hamiltonian 2.7 in two factors. One factor is periodic in the the variable Θ with a period of 2π (ring periodic) and the other is unperiodic. This is done by splitting the phase advances into average and fluctuating part:

$$\Phi_{x,y}(\Theta) = \tilde{\Phi}_{x,y}(\Theta) + \Theta\cdot Q_{x,y} \tag{3.1}$$

where $Q_{x,y}$ are the linear machine tunes.

Then we define the periodic hamiltonian functions as:

$$h_{nm\nu\mu}(\Theta) = \bar{R} \begin{pmatrix} n \\ \frac{n-\nu}{2} \end{pmatrix} \begin{pmatrix} m \\ \frac{m-\mu}{2} \end{pmatrix} \cdot a_{nm} \left(\frac{\beta_x}{2}\right)^{\frac{1}{2}} \cdot \left(\frac{\beta_y}{2}\right)^{\frac{1}{2}} \cdot e^{i\left(\nu\tilde{\Phi}_x(\Theta) + \mu\tilde{\Phi}_y(\Theta)\right)} \tag{3.2}$$

The hamiltonian can then be written as

$$H = \sum_{nm\nu\mu} h_{nm\nu\mu}\, \varepsilon_x^{\frac{n}{2}}\, \varepsilon_y^{\frac{m}{2}}\, e^{i\left(\nu\Phi_x + \mu\Phi_x + (\nu Q_x + \mu Q_y)\Theta\right)} \tag{3.3}$$

We are now looking for a canonical transformation, which removes all the parts from the hamiltonian which vary fast with Θ and retains only slowly varying and constant parts. We assume a new hamiltonian K which depends on new canonical variables J and Ψ but has a similar form to the old hamiltonian H.

$$K = \sum_{nm\nu\mu} k_{nm\nu\mu} \; J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \; e^{i\left(\nu\Psi_x+\mu\Psi_x+(\nu Q_x+\mu Q_y)\Theta\right)} \tag{3.4}$$

The new variables J,Ψ should differ only by a small relative amount from the original ones ε,Φ because the motion is dominated by the linear forces and the nonlinear forces are only distortions according to our basic assumption. Thus the canonical transformation is the identity transformation plus a small correction σ. Because the generating function removes parts of the old hamiltonian, the most obvious ansatz for σ is to assume it has the same formal dependence of the variables as H and K. As a generating function it is mixed in old and new canonical variables:

$$S(J_x,J_y,\Phi_x,\Phi_y) = J_x\Phi_x + J_y\Phi_y + \sigma\,(J_x,J_y,\Phi_x,\Phi_y,\Theta)$$

$$\sigma(\Theta) = \sum_{nm\nu\mu} \sigma_{nm\nu\mu}(\Theta)\; J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \; e^{i\left(\nu\Phi_x+\mu\Phi_x+(\nu Q_x+\mu Q_y)\Theta\right)} \tag{3.5}$$

The transformation between new and old hamiltonian is always

$$K = H + \partial S/\partial\Theta \tag{3.6}$$

and the transformation between old and new canonical variables is:

$$\Psi_{x,y} = \partial S/\partial J_{x,y} \quad ; \quad \varepsilon_{x,y} = \partial S/\partial \Phi_{x,y} \tag{3.7}$$

4. Perturbation Expansion of Old and New Hamiltonian in Mixed Canonical variables

The algorithm described in this section was developed by Moser /MOS55/. Explicit expressions for the hamiltonian up to second order and the generating function in first order have been presented by Schoch /SCH57/ and Hagedorn/HAG57/.

We insert our expressions for K and S into equation 3.6 in order to determine the functions k and σ by expressing the momentum variable ε by J and the coordinate variable Ψ by Φ. Powers of ε and exponentials of Ψ must be expanded in a taylor series:

$$\varepsilon_x^{\frac{n}{2}} \cdot \varepsilon_y^{\frac{m}{2}} = J_x^{\frac{n}{2}} \cdot J_y^{\frac{m}{2}} + \frac{n}{2} J_x^{\frac{n-2}{2}} J_y^{\frac{m}{2}} \cdot \partial\sigma/\partial\Phi_x + \frac{m}{2} J_x^{\frac{n}{2}} J_y^{\frac{m-2}{2}} \cdot \partial\sigma/\partial\Phi_y + \dots$$

$$= J_x^{\frac{n}{2}} \cdot J_y^{\frac{m}{2}}$$

$$+ i \sum_{n'm'\nu'\mu'} \frac{n\nu'}{2} \sigma_{n'm'\nu'\mu'} J_x^{\frac{n+n'-2}{2}} J_y^{\frac{m+m'}{2}} e^{i\left(\nu'\Phi_x + \mu_y'\Phi + (\nu' Q_x + \mu' Q_x)\Theta\right)}$$

$$+ i \sum_{n'm'\nu'\mu'} \frac{m\mu'}{2} \sigma_{n'm'\nu'\mu'} J_x^{\frac{n+n'}{2}} J_y^{\frac{m+m'-2}{2}} e^{i\left(\nu'\Phi_x + \mu'\Phi_y + (\nu' Q_x + \mu' Q_y)\Theta\right)}$$

$$+ \dots \qquad (4.1)$$

$$e^{i\left(\nu\Psi_x + \mu\Psi_y + (\nu Q_x + \mu Q_y)\Theta\right)} = e^{i\left(\nu\Phi_x + \mu\Phi_y + (\nu Q_x + \mu Q_y)\Theta\right)} \left(1 + \nu\frac{\partial\sigma}{\partial J_x} + \mu\frac{\partial\sigma}{\partial J_y} + \dots\right)$$

$$= e^{i\left(\nu\Phi_x + \mu\Phi_y + (\nu Q_x + \mu Q_y)\Theta\right)}$$

$$+ \sum_{n'm'\nu'\mu'} \frac{n'\nu}{2} \sigma_{n'm'\nu'\mu'} J^{\frac{n'-2}{2}} J^{\frac{m'}{2}} e^{i\left((\nu+\nu')(\Phi_x + Q_x\Theta) + (\mu+\mu')(\Phi + Q_x)\Theta\right)}$$

$$+ \sum_{n'm'\nu'\mu'} \frac{m'\mu}{2} \sigma_{n'm'\nu'\mu'} J^{\frac{n'}{2}} J^{\frac{m'-2}{2}} e^{i\left((\nu+\nu')(\Phi_x + Q_x\Theta) + (\mu+\mu')(\Phi + Q_x)\Theta\right)}$$

$$+ \dots \qquad (4.2)$$

These expressions get inserted in the equation 3.6 which relates the old and new hamiltonian to the generating function.

$$\sum_{nm\nu\mu} i(\nu Q_x+\mu Q_y)\sigma_{nm\nu\mu}+\frac{\partial\sigma_{nm\nu\mu}}{\partial\Theta}$$

$$=\sum_{nm\nu\mu}(k_{nm\nu\mu}-h_{nm\nu\mu})\,J^{\frac{n}{2}}J^{\frac{m}{2}}\,e^{i\left(\nu\Phi_x+\mu\Phi_y+(\nu Q_x+\mu Q_y)\Theta\right)}$$

$$+\,i\sum_{\substack{n'm'\nu'\mu'\\ n''m''\nu''\mu''}}\Bigg[\frac{n'\nu''}{2}\left(k_{n''m''\nu''\mu''}\sigma_{n'm'\nu'\mu'}-h_{n'm'\nu'\mu'}\sigma_{n''m''\nu''\mu''}\right)\cdot J^{\frac{n'+n''-2}{2}}J^{\frac{m'+m''}{2}}\cdot e^{i\left((\nu'+\nu'')(\Phi_x+Q_x\Theta)+(\mu'+\mu'')(\Phi_y+Q_y\Theta)\right)}\Bigg]$$

$$+\,i\sum_{\substack{n'm'\nu'\mu'\\ n''m''\nu''\mu''}}\Bigg[\frac{m'\mu''}{2}\left(k_{n''m''\nu''\mu''}\sigma_{n'm'\nu'\mu'}-h_{n'm'\nu'\mu'}\sigma_{n''m''\nu''\mu''}\right)\cdot J^{\frac{n'+n''}{2}}J^{\frac{m'+m''-2}{2}}\cdot e^{i\left((\nu'+\nu'')(\Phi_x+Q_x\Theta)+(\mu'+\mu'')(\Phi_y+Q_y\Theta)\right)}\Bigg]$$

$$+\;.....\qquad(4.3)$$

The terms are ordered according to their powers n/2 and m/2 in the J and the arguments $\nu\Phi,\mu\Phi$ of the exponentials. Because the equation holds for any value of the amplitude J or phase Φ it is true for each summand characterized by $nm\nu\mu$:

$$i(\nu Q_x+\mu Q_y)\sigma_{nm\nu\mu}+\partial\sigma_{nm\nu\mu}/\partial\Theta =$$

$$k_{nm\nu\mu}-h_{nm\nu\mu}+i\sum_{\substack{n'm'\nu'\mu'\\ n''m''\nu''\mu''}}\frac{n'\nu''}{2}\left(k_{n''m''\nu''\mu''}\cdot\sigma_{n'm'\nu'\mu'}-h_{n'm'\nu'\mu'}\cdot\sigma_{n''m''\nu''\mu''}\right)$$

$$+\,i\sum_{\substack{n'm'\nu'\mu'\\ n''m''\nu''\mu''}}\frac{m'\mu''}{2}\left(k_{n''m''\nu''\mu''}\cdot\sigma_{n'm'\nu'\mu'}-h_{n'm'\nu'\mu'}\cdot\sigma_{n''m''\nu''\mu''}\right)$$

$$+\;....\qquad(4.4)$$

The two sums stem from expanding
ε_x,Φ_x and ε_y,Φ_y

respectively and will be referred to as x-like and y-like. The eight indices of the double sum are related by :

1rst sum (x-like): $n = n'+n''-2$, $m = m'+m''$, $\nu = \nu'+\nu''$, $\mu = \mu'+\mu''$ 2nd sum (y-like): $n = n'+n''$, $m = m'+m''-2$, $\nu = \nu'+\nu''$, $\mu = \mu'+\mu''$ (4.5)

At this point, we express the periodic functions h,k,σ by their fourier coefficients

$$h^{q}_{nm\nu\mu q} = \frac{1}{2\pi} \int d\Theta \; h_{nm\nu\mu}(\Theta) \quad e^{-iq\Theta} \tag{4.6}$$

Because the relation between h,k,σ must hold for every Θ, it is true for each single fourier component of h,k and σ :

$$\sigma^{q}_{nm\nu\mu} = \frac{k^{q}_{nm\nu\mu} - h^{q}_{nm\nu\mu}}{i(\nu Q_x+\mu Q_x+q)}$$
$$+ \frac{i\sum\limits_{\substack{n'm'\nu'\mu'q' \\ n''m''\nu''\mu''}} \frac{n'\nu''}{2} \left(k^{q-q'}_{n''m''\nu''\mu''}\sigma^{q'}_{n'm'\nu'\mu'} - h^{q-q'}_{n'm'\nu'\mu'}\sigma^{q'}_{n''m''\nu''\mu''}\right)}{i(\nu Q_x+\mu Q_y+q)}$$
$$+ \frac{i\sum\limits_{\substack{n'm'\nu'\mu'q' \\ n''m''\nu''\mu''}} \frac{m'\mu''}{2} \left(k^{q-q'}_{n''m''\nu''\mu''}\sigma^{q'}_{n'm'\nu'\mu'} - h^{q-q'}_{n'm'\nu'\mu'}\sigma^{q'}_{n''m''\nu''\mu''}\right)}{i(\nu Q_x+\mu Q_y+q)}$$
$$+ \;\ldots. \tag{4.7}$$

5. Solving the equation for k and σ by Iteration

Now we require that the new hamiltonian K contains only terms which vary slowly (resonant terms) or which are constant, thus terms with:

$$\nu Q_x+\mu Q_y + q = \text{small or } \nu=\mu=q=0 \tag{5.1}$$

For such terms, the whole right hand side of equation 4.7 must vanish and we can chose σ to be zero in this case. For all other terms we can solve the equation by iteration. We start by inserting

$$\sigma^{q\,(0)}_{nm\nu\mu} = 0 \tag{5.2}$$

in the equation and obtain in first order

$$\sigma^{q\,(I)}_{nm\nu\mu} = \frac{-\,h^q_{nm\nu\mu}}{i\,(\nu Q_x+\mu Q_x+q)} \quad \text{and} \quad k^{q\,(I)}_{n_r m_r \nu_r \mu_r} = h^q_{n_r m_r \nu_r \mu_r} \tag{5.3}$$

The index r indicates resonant or constant terms. In the next iteration step we obtain :

$$\sigma^{q\,(II)}_{nm\nu\mu} = \frac{-\sum\limits_{\substack{n'm'\nu'\mu'\\ n''m''\nu''\mu''}} \frac{n''\nu'}{2}\, h^{q-q'}_{n'_r m'_r \nu'_r \mu'_r} h^{q'}_{n''m''\nu''\mu''} - \frac{n'\nu''}{2}\, h^{q-q'}_{n'm'\nu'\mu'}\, h^{q'}_{n''m''\nu''\mu''}}{i\,(\nu Q_x+\mu Q_y+q)\,(\nu'' Q_x+\mu'' Q_y+q')}$$

$$+ \frac{-\sum\limits_{\substack{n'm'\nu'\mu'\\ n''m''\nu''\mu''}} \frac{m''\mu'}{2}\, h^{q-q'}_{n'_r m'_r \nu'_r \mu'_r} h^{q'}_{n''m''\nu''\mu''} - \frac{m'\mu''}{2}\, h^{q-q'}_{n'm'\nu'\mu'}\, h^{q'}_{n''m''\nu''\mu''}}{i\,(\nu Q_x+\mu Q_y+q)\,(\nu'' Q_x+\mu'' Q_y+q')} \tag{5.4}$$

$$k^{q\,(II)}_{nm\nu\mu_r} = \sum\limits_{\substack{n'm'\nu'\mu'\\ n''m''\nu''\mu''}} \frac{\frac{n''\nu'}{2}\, h^{q-q'}_{n'_r m'_r \nu'_r \mu'_r}\, h^{q'}_{n''m''\nu''\mu''} - \frac{n'\nu''}{2}\, h^{q-q'}_{n'm'\nu'\mu'} h^{q'}_{n''m''\nu''\mu''}}{i(\nu'' Q_x+\mu'' Q_y+q')}$$

$$+ \sum\limits_{\substack{n'm'\nu'\mu'\\ n''m''\nu''\mu''}} \frac{\frac{m'\mu''}{2}\, h^{q-q'}_{n'_r m'_r \nu'_r \mu'_r}\, h^{q'}_{n''m''\nu''\mu''} - \frac{m'\mu''}{2}\, h^{q-q'}_{n'm'\nu'\mu'} h^{q'}_{n''m''\nu''\mu''}}{i(\nu'' Q_x+\mu'' Q_y+q')} \tag{5.5}$$

If we apply the canonical transformation generated by S including all terms up to second order(quadratic in h), the new hamiltonian K contains only constant terms or resonant terms up to 2nd order. The lowest order oscillating terms are third order terms (cubic in h). We cut the expansion and iteration at this point and assume that the particle motion is described sufficiently accurately by the terms up to 2nd order.

6. Introduction of a Thin Lens Approximation and Evaluation of the Greens Function

To evaluate the new hamiltonian and the generating function and express them in a closed form in terms of the linear lattice functions, we have to carry out an inverse fourier transformation to obtain the Greens function for the differential equation 3.6. It is convenient for later evaluation on a computer to assume the nonlinear forces are acting as thin lenses on the particles. This is no restriction on the generality of the result and has the advantage of dealing with sums of terms around the lattice rather than dealing with integrals. It is also straight forward to extend the result to the general case.
Thus we write for the multipole coefficients as a function of the longitudinal positions i around the lattice:

$$a_{nm}(\Theta) = \sum_i a^i_{nm}\, \delta(\Theta-\Theta_i) \quad ; \quad a^i_{nm} = \frac{1}{R}\int_{s-l_i/2}^{s+l_i/2} ds\, a_{nm}(s) \tag{6.1}$$

The fourier transform of the function h is then :

$$h^q_{nm\nu\mu} = \frac{1}{2\pi}\sum_i h^i_{nm\nu\mu} e^{i\left(\nu(\Phi^i_x - Q_x\Theta^i) + \mu(\Phi^i_y - Q_y\Theta^i) - q\Theta^i\right)} \tag{6.2}$$

$$h^i_{nm\nu\mu} = \sum_i \binom{n}{\frac{n-\nu}{2}}\binom{m}{\frac{m-\mu}{2}}\left(\frac{\beta^i_x}{2}\right)^{\frac{n}{2}}\left(\frac{\beta^i_y}{2}\right)^{\frac{m}{2}} a^i_{nm} \tag{6.3}$$

In order to carry out the sums over q and q' to obtain the Greens function, we have to evaluate sums of the form:

$$\sum_{q=-\infty}^{+\infty} \frac{e^{iq\Theta}}{\alpha+q} = \frac{\pi\, e^{-i\alpha(\Theta-\mathrm{sign}(\Theta)\pi)}}{\sin(\pi\alpha)} \quad ; \quad \lim_{\Theta\to 0}: \quad \pi\frac{\cos(\pi\alpha)}{\sin(\pi\alpha)} \tag{6.4}$$

If α is an integer or near an integer the term $q\simeq-\alpha$ gets excluded:

$$= -i(\Theta-\mathrm{sign}(\Theta)\pi)\, e^{-i\alpha\Theta} \quad ; \quad \lim_{\Theta\to 0}: \quad 0 \tag{6.5}$$

7. Evaluation of the Generating Function; First Order Terms

We are now able to evaluate the generating function S order by order at the azimuth Θ_k. The zero-th order of S is just the identity transformation. The first order terms are given by inserting eq. 6.2 into eq.5.4:

$$S^{(I)}(\Theta_k) = \sum_{nm\nu\mu} \frac{-h^q_{nm\nu\mu}}{i(\nu Q_x+\nu Q_y+q)} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} e^{i\left(\nu\Phi_x+\mu\Phi_y + (\nu Q_x+\mu Q_y+q)\Theta_k\right)}$$

$$= \sum_{inm\nu\mu} \frac{-h^i_{nm\nu\mu}}{2\pi i} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} e^{i(\nu\Phi_x+\mu\Phi_y+(\nu Q_x+\mu Q_y)\Theta_k)} \sum_q \frac{e^{i(\Theta^k-\Theta^i)}}{i(\nu Q_x+\mu Q_x+q)}$$

$$= \sum_{inm\nu\mu} \frac{-h^i_{nm\nu\mu}}{2i} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} e^{i\left(\nu(\Phi_x+\Phi_x^i) + \mu(\Phi_y+\Phi_y^i)\right)}$$

$$\begin{cases} \dfrac{e^{i\pi\cdot\mathrm{sign}(\Theta_k-\Theta_i)(\nu Q_x+\mu Q_y)}}{\sin \pi(\nu Q_x+\mu Q_y)} & \text{for } \nu Q_x+\mu Q_y \neq \text{integer} \\ -i\left(\dfrac{\Theta_k-\Theta_i}{\pi} - \mathrm{sign}(\Theta_k-\Theta_i)\right) & \text{for } \nu Q_x+\mu Q_y \simeq \text{integer} \end{cases} \tag{7.1}$$

It may appear confusing that integer and noninteger terms are distinguished after resonant terms have been excluded from the generating function in order to retain them as a driving term in the new hamiltonian. However we excluded only one term in the fourier series. All the rest of the terms 5.3 have integer but non vanishing denominators and are therefore included in the generating function. Now we want return to real numbers and combine terms with the same $|\nu+\mu|$. Then the sum over ν extends only over positive numbers the while sum over μ extends over positive and negative numbers. We find

$$S^{(I)}(\Theta_k) = \sum_{inm\nu\mu} - h^{i}_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \frac{\sin\left(\nu(\Phi_x^i+\Phi_x+ s_{ki}\pi Q_x)+\mu(\Phi_y^i+\Phi_y+ s_{ki}\pi Q_y)\right)}{\sin \pi(\nu Q_x+\mu Q_y)}$$

$$s_{ki} = \text{sign}\ (\Theta_k-\Theta_i) \tag{7.2}$$

In order to carry out the transformation between old and new coordinates, it is convenient to introduce an amplitude and a phase:

$$S^{k(I)}_{nm\nu\mu} = \sqrt{\Sigma_s^2+\Sigma_c^2} \quad ; \quad \Sigma_c = \sum_i h^i_{nm\nu\mu}\cos\left(\nu(\Phi_x^i+s_{ki}\pi Q_x)+\mu(\Phi_y^i+s_{ki}\pi Q_y)\right)$$

$$\Phi^{k(I)}_{nm\nu\mu} = \cos^{-1}\left(\frac{\Sigma_c}{S^{k(I)}_{nm\nu\mu}}\right) \quad ; \quad \Sigma_s = \sum_i h^i_{nm\nu\mu}\sin\left(\nu(\Phi_x^i+s_{ki}\pi Q_x)+\mu(\Phi_y^i+s_{ki}\pi Q_y)\right)$$

$$\tag{7.3}$$

$$S_k^{(I)} = -\sum_{nm\nu\mu} S^{k(I)}_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \frac{\sin\left(\nu\Phi_x+\mu\Phi_y+\Phi^{k(I)}_{nm\nu\mu}\right)}{\sin \pi(\nu Q_x+\mu Q_y)} \tag{7.4}$$

The same procedure for the 'integer' terms results in:

$$\Sigma_c = \sum_i h^i_{nm\nu\mu}\left(\frac{\Theta_k-\Theta_i}{\pi} - s_{ki}\right) \cos(\nu\Phi_x^i+\mu\Phi_y^i) \quad ,\text{etc} \tag{7.5}$$

$$S_k^{(I)} = \sum_{nm\nu\mu} S^{k(I)}_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \cos(\nu\Phi_x+\mu\Phi_y+\Phi^{k(I)}_{nm\nu\mu}) \tag{7.6}$$

8. Evaluation of the Generating Function; Second Order - First Part

We are turning now to the second order terms for S. There are four parts of second order terms:
There are two sums each for 'x-like' terms and 'y-like' terms respectively. The first sum in each group contains the product of a 'resonant' or 'constant' coefficient h with a 'non resonant' one. The second term in each group contains products of 'resonant' and 'non resonant' terms h as well with a 'non resonant' one. We start with the first term:

Since $\nu_r' Q_x + \mu_r' Q_y + q - q' = 0$ (or $\simeq 0$) for resonant or constant terms

$$\nu Q_x + \mu Q_y + q = (\nu' + \nu'')Q_x + (\mu' + \mu'')Q_y + q \simeq \nu'' Q_x + \mu'' Q_x + q'$$

$$q_r \equiv q - q' \tag{8.1}$$

For constant terms $\nu_r' = \mu' = q - q' = 0$ the expression vanishes because it has ν_r' as a factor so that we have to deal with resonant terms only:

$$S_{kr}^{(II)} = \sum_{\substack{nm\nu\mu \\ ij}} \sum_{\substack{n''m''\nu''\mu'' \\ n_r'm_r'\nu_r'\mu_r'}} \frac{n''\nu_r' \, h^i_{n_r'm_r'\nu_r'\mu_r'} h^j_{n''m''\nu''\mu''}}{8i\pi^2} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} e^{i\left(\nu\Phi_x + \mu\Phi_y + (\nu Q_x \mu Q_y)\Theta_k\right)}$$

$$\cdot \, e^{i\left(\nu_r'\Phi_x^i + \mu_r'\Phi_x^i + (\nu_r' Q_x + \mu_r' Q_y)\Theta^i\right)} \quad \cdot \quad e^{i\left(\nu''\Phi_x^j + \mu''\Phi_y^j + (\nu'' Q_x + \mu'' Q_y)\Theta^j\right)}$$

$$\sum_{q_r} e^{i(\Theta^k - \Theta^i)q_r} \cdot \sum_{q'} \frac{e^{-i(\Theta^k - \Theta^j)q'}}{(\nu'' Q_x + \mu'' Q_y + q')^2} \tag{8.2}$$

The sum over

$$n_r m_r \nu_r \mu_r q_r$$

extends over only a few terms. We carry out the sum over q' and find

$$S_{kr}^{(II)} = \sum_{\substack{nm\nu\mu \\ ij}} \sum_{\substack{n''m''\nu''\mu'' \\ n_r'm_r'\nu_r'\mu_r'q_r}} \frac{n''\nu_r' \, h^i_{n_r'm_r'\nu_r'\mu_r'} h^j_{n''m''\nu''\mu''} \left(\Theta^k - \Theta^j - \pi s_{kj} + \pi\cot\pi(\nu'' Q_x + \mu'' Q_y)\right)}{8\pi \, \sin\pi(\nu'' Q_x + \mu'' Q_y)}$$

$$J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \quad e^{i\left(\nu\Phi_x + \mu\Phi_y\right)} \cdot e^{i\left(\nu''\Phi_x^j + \mu''\Phi_y^j + s_{kj}\pi(\nu'' Q_x + \mu'' Q_y)\right)}$$

$$\cdot \, e^{i\left(\nu_r'\Phi_x^i + \mu_r'\Phi_x^i + (\nu_r' Q_x + \mu_r' Q_y + q_r)(\Theta^k - \Theta^i)\right)} \tag{8.3}$$

In most cases, where there are resonant terms in first order, we need not proceed with the perturbation expansion. On the other hand, usually we try to avoid isolated resonances driven by first order terms by a careful choice of the tunes. Thus we will exclude from our considerations situations where the above terms may become important. One should mention at this point, that the first sums just evaluated for the generating function vanish for the second order hamiltonian coefficients 5.5. Because we would have

$$\nu Q_x+\mu Q_y+q=0.$$

with the restriction $\nu' Q_x+\mu' Q_y+q-q'=0$, we also have $\nu'' Q_x+\mu'' Q_y+q'=0$.

For such terms however the coefficient $\sigma(n''m''\nu''\mu''q')$ in 5.3 is zero. Thus there are no contributions to second order hamiltonian coefficients k from the first sums.

9. Evaluation of Generating Function Second Order; Second Part

We move now on to the second sum 5.4. If we insert the coefficients h_i (6.3) we find for the x-like terms:

$$S_k^{(II)} = \sum_{\substack{nm\nu\mu \\ ij}} \sum_{\substack{n'm'\nu'\mu' \\ n''m''\nu''\mu''}} \frac{n'\nu'' h^i_{n'm'\nu'\mu'} h^j_{n''m''\nu''\mu''}}{8i\pi^2} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} e^{i\left(\nu\Phi_x+\mu\Phi_y+(\nu Q_x+\mu Q_y)\Theta_k\right)}$$

$$\cdot\, e^{i\left(\nu'\Phi^i_x+\mu'\Phi^i_x+(\nu' Q_x+\mu' Q_y)\Theta^i\right)} \quad \cdot \quad e^{i\left(\nu''\Phi^j_x+\mu''\Phi^j_y+(\nu'' Q_x+\mu'' Q_y)\Theta^j\right)}$$

$$\sum_q \frac{e^{i\Theta^k q}}{\nu Q_x+\mu Q_y+q} \cdot \sum_{q'} \frac{e^{i(\Theta^i-\Theta^j)q'}}{\nu'' Q_x+\mu'' Q_y+q'} \qquad (9.1)$$

Carrying out the sums over q and q' and combining complex numbers to real numbers as before leaves us with:

$$S_k^{(II)} = \sum_{\substack{nm\nu\mu ij \\ n''m''\nu''\mu'' \\ n'm'\nu'\mu'}} \frac{n'\nu'' h^i_{n'm'\nu'\mu'} h^j_{n''m''\nu''\mu''} \cos\left(\nu''|\Phi^j_x-\Phi^i_x|+\mu''|\Phi^j_y-\Phi^i_y|+\pi(\nu'' Q_x+\mu'' Q_y)\right)}{2\sin\pi(\nu Q_x+\mu Q_y)\,\sin\pi(\nu'' Q_x+\mu'' Q_y)}$$

$$\cdot J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \sin\left(\nu(\Phi_x+\Phi^i_x)+\mu(\Phi_y+\Phi^i_y)+s_{ki}\pi(\nu Q_x+\mu Q_y)\right)$$

$$(9.2)$$

For the 'y-like' terms, we get a similar result. The symmetry between x- and y-terms is only broken because ν but not μ is restricted to positive integers. This results in a factor sign(μ) for the 'y-like' terms. Besides this, the y-like terms differ from the x-like terms only by the factor $m\mu$ instead of $n\nu$ and the different relationship between n,m and n'm',n"m" for x and y like terms.

If we exclude the existence of resonant terms in first order, we don't need to exclude any terms in the above sum 9.1 over q'. Thus we will have no

$$\nu''+Q_x\mu''Q_y=\text{integer - terms.}$$

except constant terms with $\nu''=''\mu=0$ which vanish because because of the factor ν''.

The sum in 9.1 over q however contains second order denominators which in general include terms

$$\nu Q_x+\mu Q_y\simeq\text{integer}.$$

Therefore for each second order resonant term to be retained in the hamiltonian, we keep the complementary sum over q in the generating function S which has the form:

$$S_k^{(II)}=\sum_{\substack{nm\nu\mu ij\\ n''m''\nu''\mu''\\ n'm'\nu'\mu'}} \frac{n'\nu'' h^i_{n'm'\nu'\mu'} h^j_{n''m''\nu''\mu''} \cos\left(\nu''|\Phi^j_x-\Phi^i_x|+\mu''|\Phi^j_y-\Phi^i_y|+\pi(\nu''Q_x+\mu''Q_y)\right)}{2\ \sin\pi(\nu''Q_x+\mu''Q_y)}$$

$$\left(\frac{\Theta_i-\Theta_k}{\pi}+s_{ki}\right)\cdot J_x^{\frac{n}{2}}J_y^{\frac{m}{2}}\cos\left(\nu(\Phi_x+\Phi^i_x)+\mu(\Phi_y+\Phi^i_y)\right) \tag{9.3}$$

in the second order generating function.

We now define the second order coefficient

$$\sigma^{i(II)}_{nm\nu\mu}=\sum_{\substack{n''m''\\ \nu''\mu''j}} \frac{n'\nu'' h^i_{\nu'\mu'\nu'\mu'} h^j_{n''m''\nu''\mu''} \cos\left(\nu''|\Phi^j_x-\Phi^i_x|+\mu''|\Phi^j_y-\Phi^i_y|+\pi(\nu''Q_x+\mu''Q_y)\right)}{2\ \sin\ \pi(\nu''Q_x+\mu''Q_y)} \tag{9.4}$$

$$\left(\sigma^{i(I)}_{nm\nu\mu}\ =\ h^i_{nm\nu\mu}\right)$$

and see the analogy between first (see 7.2) and second order terms:

$$S^{(II)}(\Theta_k) = \sum_{inm\nu\mu} \sigma^{i(II)}_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \frac{\sin\left(\nu(\Phi_x^i - \Phi_x + s_{ki}\pi Q_x) + \mu(\Phi_y^i + \Phi_y + s_{ki}\pi Q_y)\right)}{\sin \pi(\nu Q_x + \mu Q_y)} \tag{9.5}$$

First and second order terms differ only by different coefficients σ. The dependence on the variables is the same for all orders. Of course the second order coefficients include higher orders n+m than we have in first order. The second order coefficients for each lattice point i require a sum over the whole lattice starting from i and a sum over all pair of first order terms which combine to the second order term under consideration according to the rules 4.5.

We proceed in the same way as for the first order terms by defining an amplitude and a phase

$$S^{kx,y(II)}_{nm\nu\mu} \text{ and } \Phi^{kx,y(II)}_{nm\nu\mu} \tag{9.6}$$

for x and y-like terms respectively.

The generating function up to 2nd order is therefore of the form:

$$\begin{aligned} S(\Theta_k) = J_x\Phi_x + J_y\Phi_y &+ \sum_{nm\nu\mu} -S^{k(I)}_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} e^{i\,(\nu\Phi_x + \mu\Phi_y + \Phi^{k(I)}_{nm\nu\mu})} \\ &+ \sum_{nm\nu\mu} S^{kx(II)}_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} e^{i\,(\nu\Phi_x + \mu\Phi_y + \Phi^{kx(II)}_{nm\nu\mu})} \\ &+ \sum_{nm\nu\mu} S^{ky(II)}_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} e^{i\,(\nu\Phi_x + \mu\Phi_y + \Phi^{ky(II)}_{nm\nu\mu})} \\ &+ \text{higher orders} \end{aligned} \tag{9.7}$$

It is interesting to notice that the transformation between new and old canonical momenta J and ε is essentially a fourier transform in the phase angle Φ with coefficients expressed in a closed form in terms of the multipole coefficients and the linear lattice functions.

If no resonant terms have to be retained in the hamiltonian, the transformation function, $\varepsilon = J + \partial S/\partial\Phi$ describes the whole effect of the nonlinear fields up to the order it is expanded. One can consider it as a 'distortion function'. It is a ring periodic function which describes the distortion of the beam emittance as a function of the unperturbed emittance J and the particle phase Φ. J is solution of a trivial new hamiltonian which contains only constant terms

$$K = \sum_{nm} k_{nm00} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \tag{9.8}$$

The validity of the description of the nonlinear effects by the generating function and a trivial hamiltonian is however restricted to the case where the distortions ε-J and Ψ-Φ are small, because this was an explicit demand as we truncated the taylor expansion for powers of ε and exponentials of Ψ(eq's 4.1, 4.2).

Nevertheless it is very useful to calculate the generating function. One recognizes which multipole component are important for the particle motion and it is easy to relate the strength of the distortion with lattice parameters like systematic multipole errors, phase advances etc.

Fig. A1 shows as an example the comparison between phase space trajectories obtained by tracking (solid lines) and obtained from distortion functions (dashed lines). The lattice contains just one strong sextupole represented by five kicks at a betatron phase advance spacing of $\Delta\Phi$=0.01. There is no betatron amplitude in the y-plane. The horizontal tune is 0.27.

If the amplitude doesn't exceed $\simeq$1/2 of the maximum stable amplitude represented by the outer solid trajectory, tracking and perturbation theory agree fairly well. There are strong differences in the trajectories at the stability limit. However, the outermost dashed curve is also what one can consider as a stability limit for distorted trajectories. The amplitude distortion ε-J starts to exceed at this amplitude the increase in the amplitude itself thus $\partial\varepsilon/\partial J$ is zero for this trajectory. This agreement is a very surprising and encouraging property of distortion functions. The comparison has been repeated for another tune far from a resonance Q=0.38. The result is shown in fig A2. One finds the same kind of qualitative agreement between tracking and distortion function.

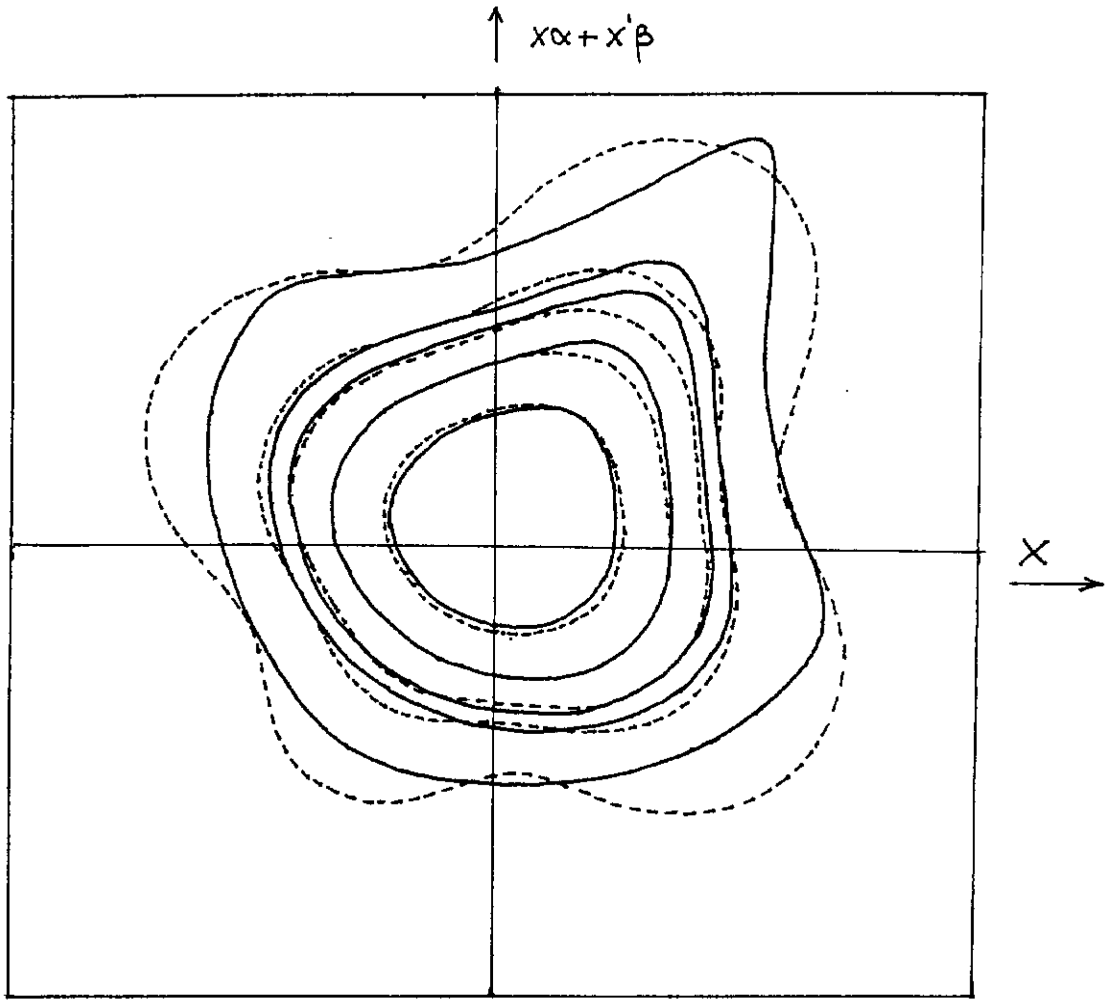

Fig A1 Comparison between Tracking and Distortion function
solid lines are tracking, dashed lines are distortion
Q = 0.28 ,see text

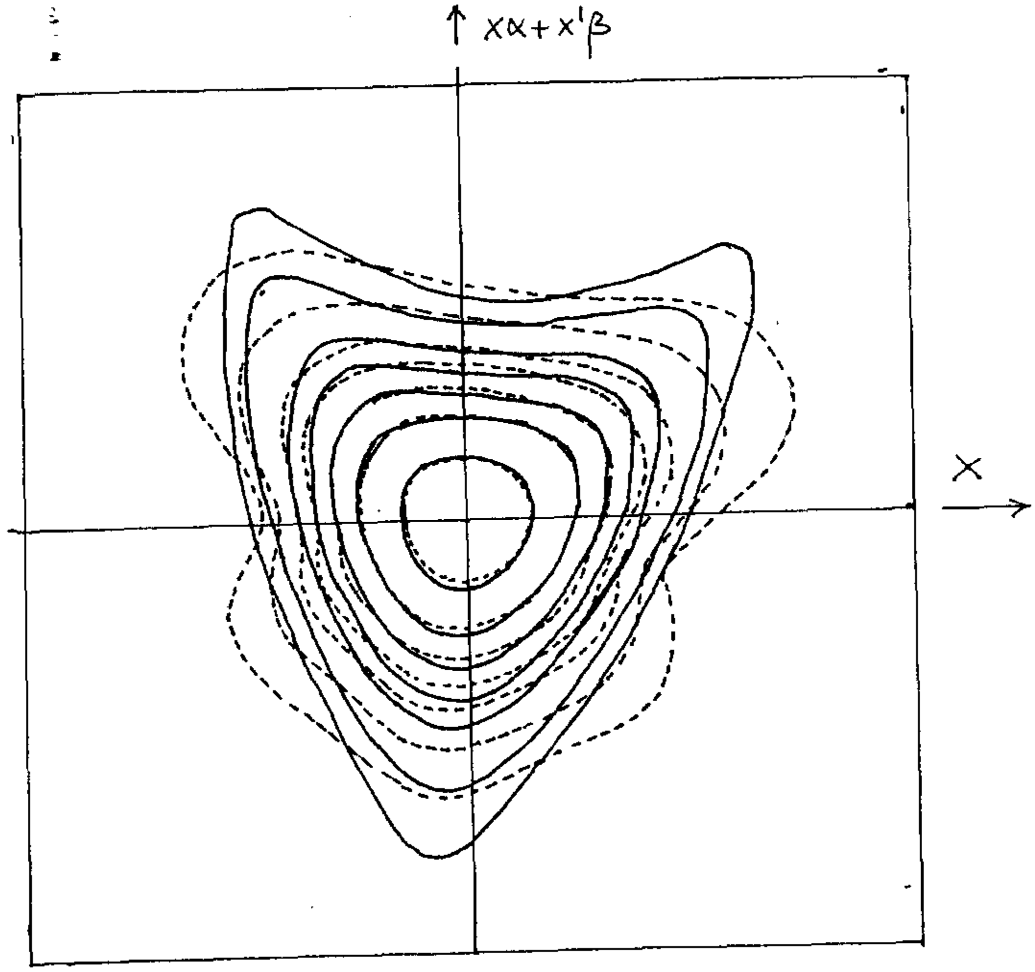

Fig A2 Comparison between Tracking and Distortion Function for a Tune of Q=0.38, see text.

10. Evaluation of the Hamiltonian; First order

We turn now to the evaluation of the new slowly varying hamiltonian based on equation 5.5 and the hamiltonian coefficients defined in eq. 6.3. According to 5.3, the first order new hamiltonian K contains just the resonant and constant parts of the old hamiltonian H. We insert eq.6.2 into equation 3.4 and combine again each term with its complex conjugate and find:

$$K^{(I)} = \frac{1}{\pi} \sum_{\substack{nm\nu\mu \\ iq}} h^{i}_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \cos\left(\nu\Phi^{i}_x + \mu\Phi^{i}_y - (\nu Q_x + \mu Q_y + q)\Theta^{i} + \nu\Psi_x + \mu\Psi_y + (\nu Q_x + \mu Q_y + q)\Theta\right) \tag{10.1}$$

The sum over $nm\nu\mu q$ extends over resonant terms only.

As for the generating function, we form an amplitude and a phase by:

$$K^{q(I)}_{nm\nu\mu} = \sqrt{\Sigma_c^2 + \Sigma_s^2} \quad ; \quad \Sigma_c = \frac{1}{\pi} \sum_{nm\nu\mu} h^{i}_{nm\nu\mu} \cos\left(\nu\Phi^{i}_x + \mu\Phi^{i}_y - (\nu Q_x + \mu Q_y + q)\Theta^{i}\right) \; ;\text{etc} \tag{10.2}$$

The hamiltonian can then be expressed in closed form:

$$K^{(I)} = \sum_{nm\nu\mu} K^{q(I)}_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \cos\left(\nu\Psi_x + \mu\Psi_y + (\nu Q_x + \mu Q_y + q)\Theta + \Phi^{q(I)}_{nm\nu\mu}\right) \tag{10.3}$$

If there is only one resonant term $nm\nu\mu$, one usually introduces new angle variables

$$\phi_x = \Psi_x + \nu\left(\frac{\nu Q_x + q}{\nu^2 + \mu^2}\right)\Theta \qquad \phi_y = \Psi_y + \mu\left(\frac{\mu Q_y + q}{\nu^2 + \mu^2}\right)\Theta \tag{10.4}$$

which are generated by the generating function:

$$F(I_x, \Phi_x, I_y, \Phi_y, \Theta) = I_x\phi_x(\Phi_x, \Theta) + I_y\phi_y(\Phi_y, \Theta) \; ; \qquad I_x = J_x; \quad I_y = J_y \tag{10.5}$$

The corresponding hamiltonian W does no longer depend explicitly on the independent variable Θ and is therefore constant:

$$W = K + \partial F/\partial\Theta = \nu J_x\left(\frac{\nu Q_x+q}{\nu^2+\mu^2}\right) + \mu J_y\left(\frac{\mu Q_y+q}{\nu^2+\mu^2}\right)$$

$$+ k_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \cos(\nu\phi_x+\mu\phi_y+\Phi^{(I)}_{nm\nu\mu}) \qquad (10.6)$$

Phase space trajectories $J(\phi)$ are given for each value of W by inverting $W_o = W(J,\phi)$ with respect to J. The separatrix is the orbit which passes trough the fix points given by $\partial W/\partial J = \partial W/\partial\phi = 0$.

11. Evaluation of the Hamiltonian Second order

The evaluation of the second order hamiltonian is much like the evaluation of the second order generating function. We again insert the hamiltonian coefficients from equation 6.3 into the expression 5.4. We already pointed out in section 8. that there is no contribution from the first part of 5.5 which involves products with resonant first order coefficients. For the second part, after carrying out the sum over q', we obtain :

$$K^{(II)} = -\sum_{\substack{in'm'\nu'\mu' \\ jn''m''\nu''\mu''}} \frac{n'\nu'' h^i_{n'm'\nu'\mu'} h^j_{n''m''\mu''\mu''}}{8\pi^2 \sin\pi(\nu''Q_x+\mu''Q_y)} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} e^{i\left(\nu\Psi_x+\mu\Psi_y+(\nu Q_x+\mu Q+q)\Theta\right)}$$

$$e^{i\left(\nu\Phi^i_x+\mu\Phi^i_y-(\nu Q_x+\mu Q_y)\Theta_i\right)} \cdot e^{i\left(\nu''(\Phi^j_x-\Phi^i_x)+\mu''(\Phi^j_y-\Phi^i_y)+s_{ji}\pi(\nu''Q_x+\mu''Q_y)\right)}$$

$$(11.1)$$

We combine again complex numbers to real numbers and ν is restricted to positive integers again. The sum over j n'm'ν'μ' and n"m"ν"μ" is the same as in equation 9.3. Thus we can use the second order coefficient σ defined in eq. 9.4 to express the generating function and we write the second order hamiltonian:

$$K^{(II)} = -\frac{1}{\pi} \sum_{\substack{nm \\ \nu\mu i}} \sigma^{i(II)}_{nm\nu\mu} J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \cos\left(\nu\Psi_x+\mu\Psi_y+(\nu Q_x+\mu Q_y+q)\Theta+\nu\Phi^i_x+\mu\Phi^i_y-(\nu Q_x+\mu Q_y)\Theta_i\right)$$

$$(11.2)$$

For the y-like terms we have a similar expression which differs only by the factor $m\nu$ in the σ-coefficient and by the factor sign(μ) for the reason pointed out in section 9 discussing the generating function. For the hamiltonian too, the second order terms have the same form as the first order term differing only by the coefficient h vs σ. Defining amplitude and phase the same way as before (see eq's 10.2, 7.3, 7.4), we obtain the new hamiltonian containing only slowly varying terms up to 2nd order in the multipole fields:

$$K = \sum_{nm\nu\mu} K^{q(I)}_{nm\nu\mu} \; J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \cos\left(\nu\Psi_x+\mu\Psi_y+(\nu Q_x+\mu Q_y+q)\Theta+\Phi^{q(I)}_{nm\nu\mu} \right)$$

$$- \sum_{nm\nu\mu} K^{qx(II)}_{nm\nu\mu} \; J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \cos\left(\nu\Psi_x+\mu\Psi_y+(\nu Q_x+\mu Q_y+q)\Theta+\Phi^{q(II)}_{nm\nu\mu} \right)$$

$$- \sum_{nm\nu\mu} K^{qy(II)}_{nm\nu\mu} \; J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \cos\left(\nu\Psi_x+\mu\Psi_y+(\nu Q_x+\mu Q_y+q)\Theta+\Phi^{q(II)}_{nm\nu\mu} \right)$$

(11.3)

Note that not all second order terms which appear in the generating function are potential driving terms in the second order hamiltonian. If the second order term in the generating function is composed of just one pair of first order terms with $\nu=\nu'+\nu''$, $\nu'=\nu''$ (same for μ), the resonance denominator is cancelled as it has been pointed out by L.Michelotti/MIC85/. Thus the transformation contribution from such terms does not get infinitely large when approaching the resonance but remains confined to off resonance values. That means for example that in second order perturbation expansion sextupole fields don't excite the 6th integer resonance (3+3) but excite only the 4th and 2nd integer resonances.

Figs A3,A4,A5 show as an example the phase space trajectories near the 4th-integer resonance driven in 2nd order by sextupoles. Just one oscillation plane in phase space is assumed. The tunes are Q=0.255(fig A3), Q=0.26 (fig A4) and Q=0.27 (fig A5). The other parameters determining the phase space trajectories were: β=100m, Θ=30mr, r°=1 inch and $b_2 \cdot 10^4 = 100$. The lines in the figures are the perturbation theory trajectories and the dots are the result of tracking.

Close to the resonance, the agreement between tracking and theory is almost perfect. The only difference is a small rotation of the theoretical trajectories with respect to the tracking result. This is due to the missing higher order (than 2) detuning terms. For the tune . Q=0.26, the agreement is still satisfying. At the largest tune disagreements become bigger and the single resonance approach starts to break down.

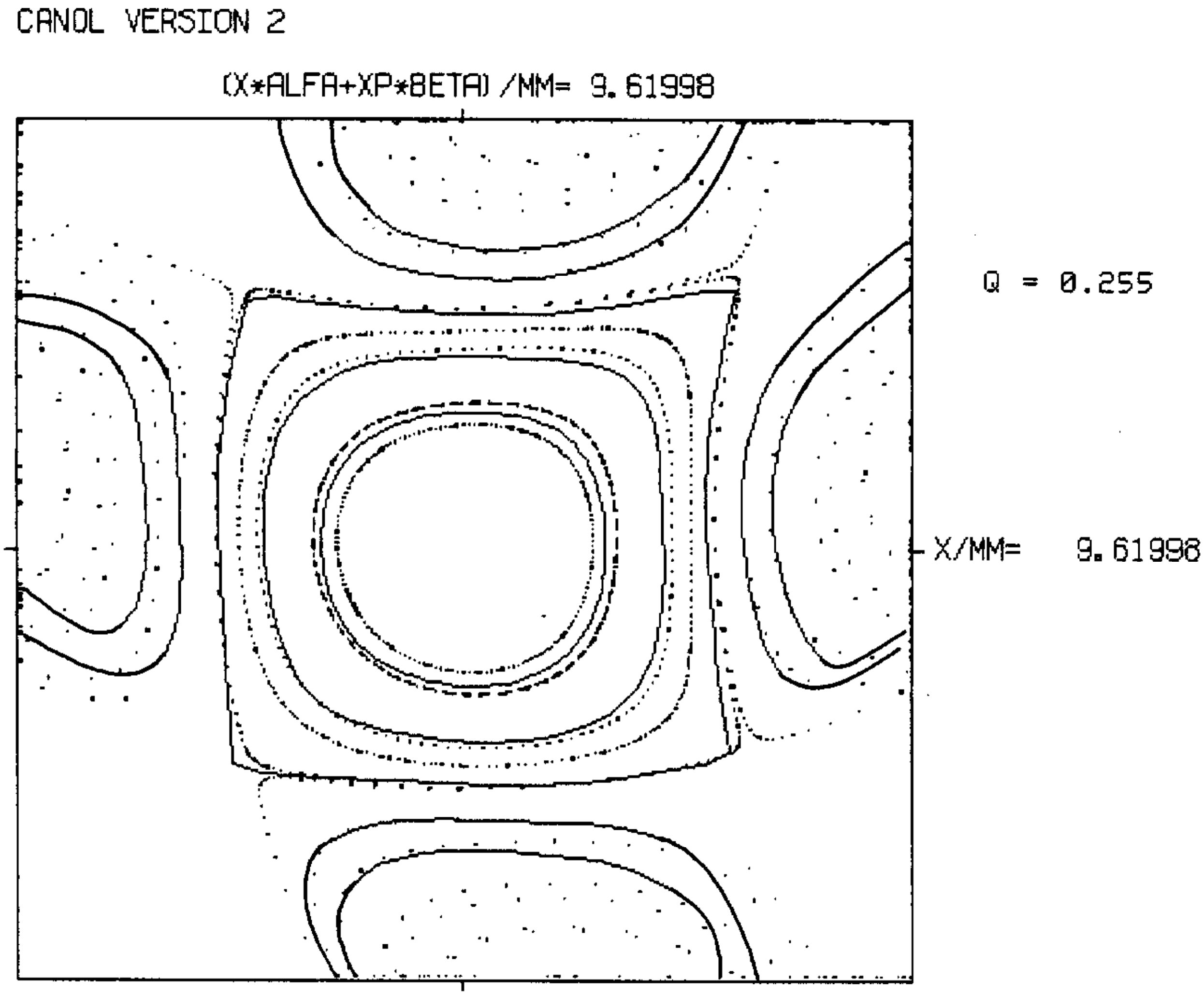

fig A3 Comparison between theoretical phase space trajectories and tracking near the sextupole excited 4th integer resonance (Q=0.255, see text)
Lines: Perturbation Theory
Dots: Tracking

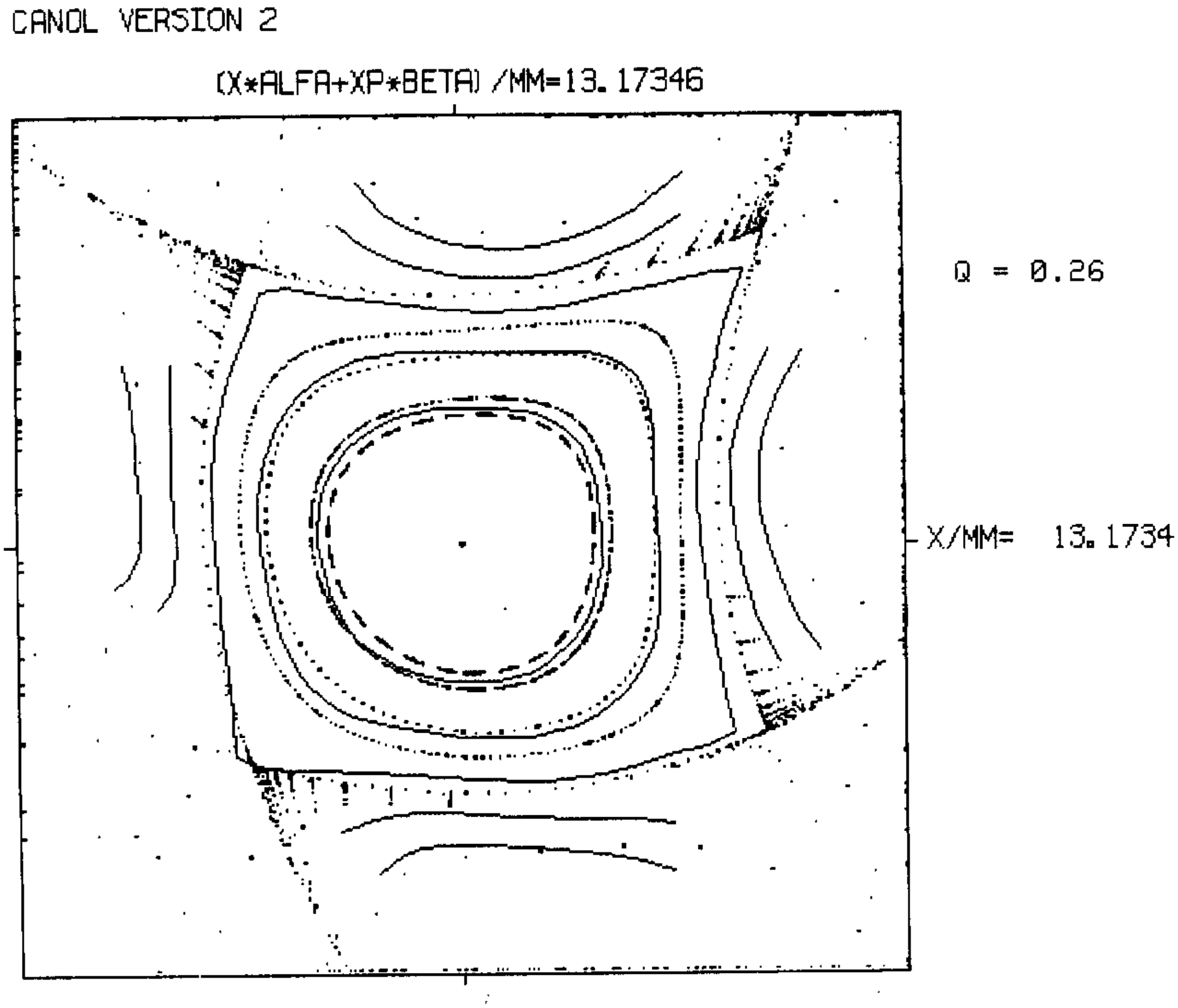

fig A4 Phase Space Trajecories for a tune of 0.26.
Lines :perturbation theory
dots: Tracking

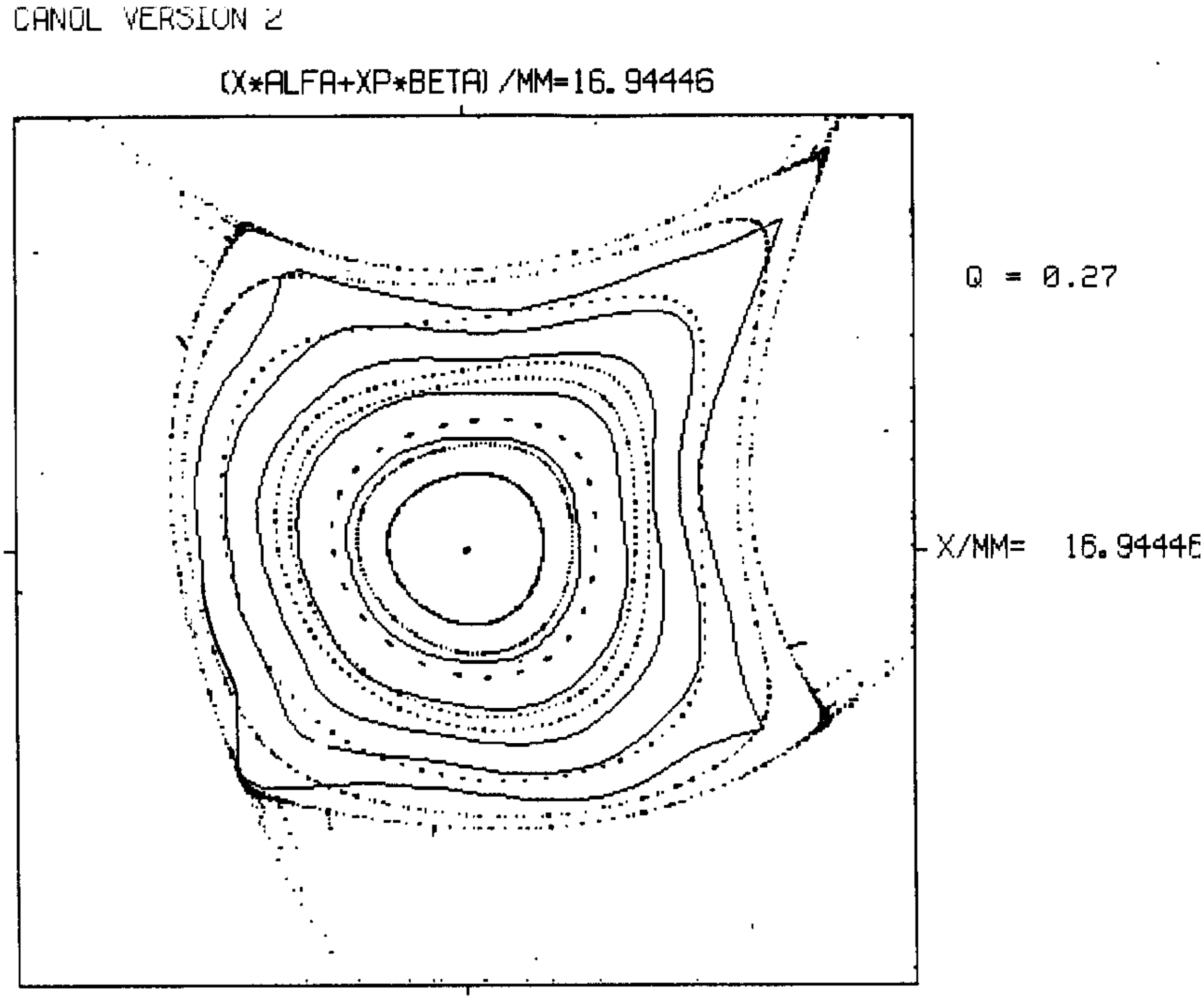

fig A5 Phase Space Trajecories for a tune of 0.27.
Lines :perturbation theory
dots: Tracking

12. Third and Higher Order Terms

It is very straight forward but a little tiresome to extend the expansion to higher than 2 orders. Higher order Taylor expansion terms have to be included in equation 4.3 and the iteration of equation 4.7 has to be continued until all terms up to the particular order are included in equations 5.4 and 5.5. The procedures to obtain the generating function and the hamiltonian are the same as before. We will not bore the reader by repeating it again and give the result for the third order generating function instead:
In third order, we have three different terms which we will characterize by xy-like, xx-like and yy-like. They differ by an integer factor f. As the two second order terms they differ by relationship between the indices of the first order terms they are created from and the indices of the third order term. The relationships are

xy-like	xx-like	yy-like	
$n=n'+n''+n'''-2$	$n=n'+n''+n'''-4$	$n=n'+n''+n'''$	
$m=m'+m''+m'''-2$	$m=m'+m''+m'''$	$m=m'+m''+m'''-4$	
$\nu=\nu'+\nu''+\nu'''$	$\nu=\nu'+\nu''+\nu'''$	$\nu=\nu'+\nu''+\nu'''$	
$\mu=\mu'+\mu''+\mu'''$	$\mu=\mu'+\mu''+\mu'''$	$\mu=\mu'+\mu''+\mu'''$	(12.1)

The integer factors (which are $n\nu$ for the x-like 2nd order terms and $m\mu$ for the y-like 2nd order terms) are far more complicated for the third order terms:

xy-term: $f_{xy}= n'(\nu-\nu')(m-m')\mu''+m'(\mu-\mu')(n-n')\nu''-n'm'\nu''\mu''$

xx-term: $f_{xx}= n'(\nu-\nu')(n-n')\nu''-n'(n'-2)\nu''\nu''$

yy-term: $f_{yy}= m'(\mu-\mu')(m-m')\mu''-m'(m-2)\mu''\mu''$

(12.2)

The third order generating function evaluated at position p in the lattice then has the form:

$$\sigma^{vw(III)}_{nm\nu\mu} = \sum_{\substack{n'm'\nu'\mu' \\ n''m''\nu''\mu'' \\ n'''m'''\nu'''\mu'''}} \frac{f_{vw}}{32} \; J_x^{\frac{n}{2}} J_y^{\frac{m}{2}} \; e^{i\left(\nu\Phi_x+\mu\Phi_y\right)}$$

$$\sum_i h^i_{n'm'\nu'\mu'} \; \frac{e^{i\left(\nu'\Phi^i_x+\mu'\Phi^i_y+s_{pi}\pi(\nu'Q_x+\mu'Q_y)\right)}}{\sin\pi\,(\nu Q_x+\mu Q_y)}$$

$$\sum_j h^j_{n''m''\nu''\mu''} \; \frac{e^{i\left(\nu''(\Phi^j_x-\Phi^i)+\mu''(\Phi^j_y-\Phi^i_y)+s_{ji}\pi(\nu''Q_x+\mu''Q_y)\right)}}{\sin\pi\,(\nu''Q_x+\mu''Q_y)}$$

$$\sum_k h^k_{n'''m'''\nu'''\mu'''} \; \frac{e^{i\left(\nu'''(\Phi^k_x-\Phi^i_x)+\mu'''(\Phi^k_y-\Phi^i_y)+s_{ki}\pi(\nu'''Q_x+\mu'''Q_y)\right)}}{\sin\pi\,(\nu'''Q_x+\mu'''Q_y)} \qquad (12.3)$$

The most remarkable and important aspect of this result is that the sum over k does not depend on the index j but on the index i. That means that for the third order expressions we don't have to carry out a triple sum but two double sums. The same is expected for any higher order. Therefore it is not impossible to evaluate the distortion function or the hamiltonian for higher orders perturbation expansion. If there is a fixed maximum resonance number $\nu+\mu$ up to which the terms in each perturbation step are calculated, the computing time increases only linearly with the expansion order.

13. The Case of a Simple Regular Lattice

If the lattice consists of a regular FODO cell structure with systematic multipole errors of the dipole magnets and of an insertion with no nonlinear fields, the driving terms and the distortion function can be expressed in terms of the phase advance per FODO cell. This can be done for any distribution of nonlinear field in the FODO cell. As an example the result for the case with just two nonlinear kicks in the middle of each half cell is presented. The phase advance per FODO cell will be denoted by Φ_c and the phase advance between two nonlinear kicks is Φ_f or Φ_d for a focusing or a defocusing quadrupole in between respectively. Horizontal and vertical phase advances and lattice functions are assumed to be the same at the positions of the nonlinear lens. The number of the regular cell is denoted by k. The situation is sketched in fig A6.

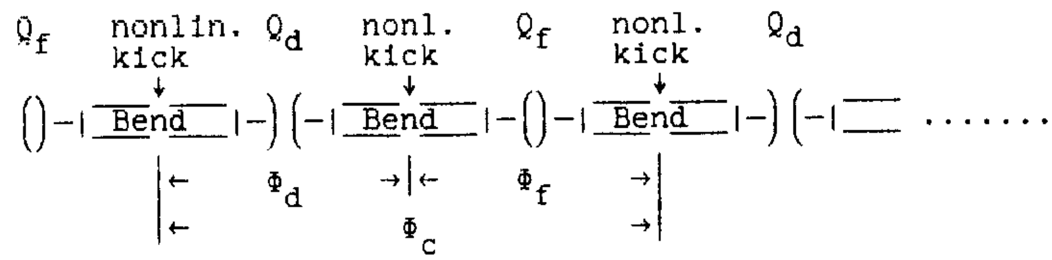

Fig. A6 Schematic view of Regular Cell structure

For this case the sum 7.3 for the first order generating function or distortion function is:

$$\Sigma_c = h_{nm\nu\mu} \sum_{k=1}^{n} \left(\sin\left(k(\nu+\mu)\Phi_c - \pi(\nu Q_x+\mu Q_y)\right) + \sin\left(k(\nu+\mu)\Phi_c + \nu\Phi_f + \mu\Phi_d - \pi(\nu Q_x + \mu Q_y)\right) \right)$$

$$= \frac{2h_{nm\nu\mu} \sin\left(\frac{k}{2}(\nu+\mu)\Phi_c\right)\cdot\cos\left(\frac{\nu\Phi_f+\mu\Phi_d}{2}\right)\cdot\cos\left(\frac{k+1}{2}(\nu+\mu)\Phi_c + \frac{\nu\Phi_f+\mu\Phi_d}{2} - \pi(\nu Q_x+\mu Q_y)\right)}{\sin(\nu+\mu)\Phi_c/2)}$$

$$\Sigma_s = \frac{2h_{nm\nu\mu} \sin\left(\frac{k}{2}(\nu+\mu)\Phi_c\right)\cdot\cos\left(\frac{\nu\Phi_f+\mu\Phi_d}{2}\right)\cdot\sin\left(\frac{k+1}{2}(\nu+\mu)\Phi_c + \frac{\nu\Phi_f+\mu\Phi_d}{2} - \pi(\nu Q_x+\mu Q_y)\right)}{\sin((\nu+\mu)\Phi_c/2)}$$

(13.1)

$$S^{I}_{nm\nu\mu} = \frac{2h_{nm\nu\mu} \sin\left(\frac{k}{2}(\nu+\mu)\Phi_c\right)\cdot\cos\left(\frac{\nu\Phi_f+\mu\Phi_d}{2}\right)}{\sin((\nu+\mu)\Phi_c/2)}$$

$$\Phi^{I}_{nm\nu\mu} = \frac{k+1}{2}(\nu+\mu)\Phi_c + \frac{\nu\Phi_f+\mu\Phi_d}{2} - \pi(\nu Q_x+\mu Q_y) \qquad (13.2)$$

For a quick estimate one can assume $\Phi_f=\Phi_d$. The amplitude of the generating function is than given by the simple expression:

$$S^I_{nm\nu\mu} = \frac{h_{nm\nu\mu}\sin\left(\frac{k}{2}(\nu+\mu)\Phi_c\right)}{\sin((\nu+\mu)\Phi_c/4)} \tag{13.3}$$

One expects a large contribution to the phase space distortion from those terms for which the argument of the sin-function in the denominator of 13.3 is equal or close to $(2k+1)\cdot\pi$ (k integer). Then the lattice sum results in a factor n for the amplitude S. Note that this is always the case for detuning terms $\nu=\mu=0$.

Unfortunately the expressions for the second order coeficients are rather complex. We first introduce the abreviations:

$$r = \nu+\mu;\ p = \pi(\nu Q_x+\mu Q_y)\ ;\ d = \nu\Phi_f+\mu\Phi_d;\ q = p + d/2 + r\Phi/2;\ \alpha=n,m,\nu,\mu \tag{13.4}$$

Reference point for the amplitude of the second order generating function S is the first element in the structure. One obtains:

$$S^{IIx}_\alpha = \sum_{\alpha'\alpha''} \frac{n'\nu''\ h_{\alpha'}h_{\alpha''}\cos(d''/2)\cos(d/2)}{2\ \sin(p)\ \sin(p'')\ \sin(r''\Phi_c/2)}$$

$$\left[\frac{\sin(\frac{k}{2}r''\Phi_c-p''-\frac{d''}{2})}{2}\left(\frac{\sin(\frac{k}{2}(r+r'')\Phi_c)}{\sin(\frac{r+r''}{2}\Phi_c)} + \frac{\sin(\frac{k}{2}(r-r'')\Phi_c)}{\sin(\frac{r-r''}{2}\Phi_c)}\right) + \frac{2\sin(q'')\sin(\frac{k}{2}r\Phi_c)}{\sin(r\Phi_c/2)}\right] \tag{13.5}$$

One recognizes that the build up of second order coefficients over the lattice is maximum if :

$$(\nu+\mu)\cdot\Phi_c/2 = j\pi \quad (\text{j integer})$$

$$(\nu''+\mu'')\Phi_c/2 = j\pi$$

$$(\nu+\nu''+\mu+\mu'')\Phi_c/2 = j\pi$$

$$(\nu-\nu''+\mu-\mu'')\Phi_c/2 = j\pi$$

The use of these formulae saves an immense amount of computing effort. It may be the only way to use the distortion function concept for very large accelerators.

APPENDIX B

Multipole Coefficients a_{nm}

Multipole coefficients as a result of a measurement are usually expressed as the relative field error measured at a certain radius r. The multipole field strength in terms of these coefficients a_k and b_k and the bend angle $\Theta°$ is given by

$$\frac{e}{p \cdot c} \int ds \left(B_y + iB_x \right) = \Sigma_k \; \Theta° (b_k + ia_k) \cdot (x+iy)^k \; / \; r^k$$

In this report , the vector potential A of the magnetic field is expanded in multiploles using coefficients a_{nm} :

$$\frac{e}{p \cdot c} \int ds \; \vec{A} \quad = \quad \begin{pmatrix} 0 \\ 0 \\ \Sigma \; a_{nm} \cdot x^n y^m \end{pmatrix} \quad \begin{matrix} \text{horizontal} \\ \text{vertical} \\ \text{longitudinal} \end{matrix} \quad \text{component}$$

The coefficients a_{nm} are given in terms of the a_k and b_k:

m = even, "normal multipole"

$$a_{nm} = - (-1)^{m/2} \frac{(n+m-1)!}{n!m!} b_{n+m-1} \cdot \Theta° \cdot r^{-(n+m-1)}$$

m = odd, "skew multipole"

$$a_{nm} = (-1)^{(m-1)/2} \frac{(n+m-1)!}{n!m!} a_{n+m-1} \cdot \Theta° \cdot r^{-(n+m-1)}$$

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